Longevity risk in life insurance products

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Longevity Risk in Life Insurance Products

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Preface

This Ph.D. thesis is comprised of five papers I have written during my Ph.D. at Tilburg University. Chapter 2 is titled “Longevity risk” and based on the paper which is published in *De Economist*, Chapter 4 is titled “The effect of product design” and published in *Insurance: Mathematics and Economics*. These two chapters have been co-authored by Anja De Waegenaere and Bertrand Melenberg. Chapter 3 is titled “Calculating Capital Requirements” and chapter 5 is titled “Hedge effects in a portfolio of life insurance products with investment risk” and are both joint work with Anja De Waegenaere and Bertrand Melenberg. Chapter 6 is my job market paper titled “Annuity decisions with systematic longevity risk”.

This Ph.D. thesis is the fruit of three years of research at the Department of Econometrics & OR at Tilburg University. It is my pleasure to take, in the remainder of this Preface, the opportunity to thank the people who made this thesis possible. First and foremost, this thesis would not have been possible without the help of my supervisors Anja De Waegenaere and Bertrand Melenberg. They started to supervise me during my master thesis, also supervised my M.Phil thesis, and as a continuation of this pleasant collaboration they invited me as a Ph.D. student on the Netspar project *Living longer in good health*. The effort and time which they devoted to me is deeply appreciated. The valuable comments I received during the weekly meetings on Tuesday afternoon has played a major role in the formation of me into a scientist. Their guidance was also pleasant which, among other things, made the time during writing the Ph.D. thesis a good time and made me decide to proceed an academic career.

I am very grateful to the committee members for the time and effort they have all put into scrutinizing my work and offering their feedback. Their valuable comments improved this thesis. I would like to thank Professors Enrico Biffis and Ralph Koijen

who were able to travel from abroad to attend the defence ceremony. I am grateful to Theo Nijman and Johan Mackenbach, for their time spent on giving feedback on my work, not only as part of the defence, but throughout my Ph.D.. The organizations CentER graduate school and the Department of Econometrics & OR at Tilburg University were a great help in completing this dissertation.

My Ph.D. research project was part of a larger project called "*Living longer in good health*", which was funded by the Network for Studies on Pensions, Aging and retirement (Netspar). The project aims to find solutions that postpone morbidity and/or its consequences (functional decline, disability, dependence on health- and social care services) until a later moment in the human lifespan. Besides me and my supervisors, the participating researchers in the project consisted of health economists and epidemiologists at Erasmus University. I like to thank my colleagues at this theme: the theme coordinator Johan Mackenbach, and senior researchers Pieter van Baal, Werner Brouwer, Job van Exel, Marc Koopmanschap, Anton Kunst, Wilma Nusselder, Suzanne Polinder, Henning Tiemeijer, and the fellow Ph.D. students Bart Klijs, Istvan Majer, Claudine de Meijer, David Rappange, and Stefan Walter. The meetings at Erasmus University broaden my knowledge on the latest research with respect to possible interventions to delay morbidity and/or its consequences, potential effects of health policy strategies on morbidity and its societal consequences in aging populations, and the development of innovative approaches to finance health care and pension systems which help to alleviate the societal consequences of the aging of populations. Whereas this thesis focus on the consequence of living longer for pension funds and insurers, longevity risk is not only a risk, but also an opportunity to live longer in good health. Moreover, not only pension funds are affected by changes in (healthy) life expectancy, but also the government and the health care system. It was a privilege for me to also see other (positive) sides of an increase in longevity than only the (negative) actuarial one during my Ph.D..

During my Ph.D. Netspar was not only visible through financing the research project, but it also provided me with a simulating research environment. My research benefited a lot from Netspar's research activities. My work was presented once at a pension day, twice at a Netspar conference, and three times at a seminar in the Netspar seminar series. I thank the people listening to my presentations and their useful comments on it. In particular I want to thank Theo Nijman for his many comments on my research he gave to me after my presentations. Many times I made use of the opportunity to go to one of Netspar's research activities, which contributed to my evolution to a researcher. I benefited a lot from stimulating discussions with other Netspar colleagues in the field of pensions. Among many others these include Rob Alessie (University of Groningen), Johannes Binswanger (Tilburg University), Lans Bovenberg (Tilburg University), Peter Broer (Netherlands Bureau for Economic Policy Analysis, CPB), Ben Heijdra (University of Groningen), David Hollanders (Tilburg University), Bas Jacobs (Erasmus University), Antoon Pelsser (Maastricht University), Michel Vellekoop (University of Amsterdam), and Marno Verbeek (Erasmus University).

I would like to express my gratitude towards Tilburg University, in particular the department of Econometrics & OR. Let me start with thanking my roommates during my Ph.D., Willemien Kets (during the first half year of my Ph.D.) and Marloes Gerichhausen (during the remainder of my Ph.D.). Although we had different research interests (i.e., their research was in the field of game theory), it was pleasant to share an office with them. I want to thank all colleagues at Tilburg Unversity, of which sone of them I want to thank in particular. From my second year onwards Kim Peijnenburg (moved from the finance department) and Lisanne Sanders (started her Ph.D.) shared a room at the same floor as me. Many times I visited their room, either for a social talk or to talk about research, which I greatly appreciated. The workplace would not be as enjoying without fellow Ph.D. students Mohammed Chahim, Salima Douhou, Thijs van der Heijden, John Kleppe, Edwin Lohmann, Roel Mehlkopf, and Chris Müris. Part of my Ph.D. obligations was to teach courses to the students at Tilburg Univesity, I would like to thank Marieke Quant for her guidance with respect to teaching. I want to thank Ruud Hendrickx for his help with LaTeX, the software program which was used to write this thesis and used for creating the slides of my presentations. I want to thank the colleagues Roger Laeven, Martin Salm, Bas Werker, Katie Carman, Joost Driessen for their helpful comments.

One of the great advantages of being a Ph.D. student is the possibility to visit (foreign) conferences. I took gratefully this opportunity by visiting them in Dalian (China), Samos (Greece), Istanbul (Turkey), Munich (Germany), New York (USA), Paris (France), Atlanta (USA), London (U.K.), Zürich (Switzerland), Toronto (Canada), and Sydney (Australia). I enjoyed visiting the cities, but even more the interesting presentations and the nice people I met. In particular, I enjoyed presentations of and conversations with Daniel Bauer (Georgia State University), Enrico Biffis (Imperial College London), David Blake (Cass Business School), Andrew Cairns (Heriot-Watt University), and Richard MacMinn (Illinois State University), all top researchers in the field of longevity risk.

Let me end this Preface by, on a more personal note, thanking my friends and family members. Without their pleasant distraction from working on my thesis after working hours it would not be impossible to put as much effort in the job as I did in order to complete this thesis in three years. I would like to thank my basketball friends at the basketball clubs TSBV Pendragon and High Five. I want to thank my housemates Tom de Groot, Noud Janssen, Marieke Sier, Hendrik Wismans, Jeffrey Lemm, Daan Stapelbroek, Thomas Slegers, Maarten van den Tweel, Marije van Liere, Tim Donckers, Viktor Van Elzakker, Shiron Jacobs, Jimmie Jacobs, and Ruen Offerman. They kept me connected with the “real world” during the past few years. Finally, I have to thank my mother, my brothers (Pascal, Armand, and Youri) and my grandparents for the guidance in my live. Without them it would have been impossible for me to even start a Ph.D., let stand alone to finalize it.

Ralph Stevens

Sydney, October 2010

Chapter 1

Introduction

The life expectancy has seen a steady increase in most of the western world over the past one and a half century. For example, the expected remaining lifetime of a Dutch male aged 65 increased from 11 years in 1875 to 12 years in 1900 to 12.5 years in 1925 to 14 years in 1950 and then decreased to 13.5 years in 1975 and thereafter again increased to almost 17.5 years in 2009.[1] The potential effects of trends in life expectancy on the value of pension liabilities present significant challenges for governments as well as individual pension funds and life insurers. Not in the trend itself, but in the fact that the future development of the life expectancy is uncertain is the major challenge. Indeed, although the past trends suggest that further increase in life expectancy is to be expected, there is considerable uncertainty regarding the future development of life expectancy. We refer to *longevity risk* as the uncertainty regarding the future development of mortality. Nowadays pension expects also recognize that longevity risk is an important risk factor. In the Global Pension Survey many pension experts in the Netherlands, United Kingdom, and Switzerland mention longevity risk as one of their top external concerns, which can be observed from Table 1.1.

[1] Source: Statistics Netherlands, http://statline.cbs.nl/.

Table 1.1: **Top external concerns**

Concern	NL	UK	DE	SZ	CZ	HR	BE
Interest rate risk	73%	83%	67%	50%	50%	0%	100%
Stability of the financial system	53%	17%	67%	60%	50%	100%	33%
Market volatility	33%	50%	67%	70%	50%	100%	0%
Longevity risks	67%	67%	33%	60%	0%	0%	0%
Performance of investment managers	20%	17%	0%	30%	0%	0%	0%
International regulation framework	0%	0%	67%	20%	50%	0%	67%
Domestic regulation	13%	17%	0%	10%	100%	50%	33%
Counterparty risk	0%	0%	0%	0%	0%	50%	33%
Inflation risk	33%	17%	0%	0%	0%	0%	33%

This table displays the percentage of respondents who place this item in the top 3 to the question: "What are the top three external concerns facing your pension fund?". The results are given for the Netherlands (NL), United Kingdom (UK), Germany (DE), Switzerland (SZ), Chech Republic (CZ), Croatia (HR), and Belgium (BE). Global Pension Survey (GPS) polls high-level pension experts of pension funds around the world to gauge recent developments, trend data, investment plans, and the prospects of individual pension funds and the economy at large. Source: GPS, third quarter 2010.

In the Netherlands, pension funds are required to value their assets and liabilities using market values since 2007. Pension funds typically have long term obligations to their participants. The funding ratio, i.e., the value of the assets divided by the value of the liabilities, is an indication whether a pension fund has enough wealth to meet its future obligations. The uncertainty in the funding ratio of pension funds became visible after the financial crisis in 2008. In 2008, pension funds in the Netherlands faced large losses of 36.7% on their equity portfolio. The equity market recovered from the dip in 2008, with a return of 31.8% on equity in 2009.[2] After a recovery in the equity market, the pension funds were faced with low interest rates, for example the 10 years interest rate has decreased from 5.027% on June 30, 2008 to 2.377% on August 31, 2010.[3] Because the regulator requires pension funds to calculate the value of the liabilities using the swap rate to discount future payments, this is the main reason of the decrease in the funding ratio of pension funds in the Netherlands. In the first quarter of 2010 in the Netherlands 122 pension funds (2,436,730 participants) had a funding ratio of less than 105%, 211 pension

[2]Source: Dutch Central Bank (DNB), Table T8.5.
[3]Source: Dutch Central Bank (DNB), Table T1.3.

funds (2,973,562 participants) had a funding ratio of between 105% and 130% and only 28 pension funds (72,847 participants) had a funding ratio of more than 130%.[4]

In August 2010, while many pension funds already having a low funding ratio, the actuarial society in the Netherlands ("Actuarieel Genootschap") published a revised version of the future best estimates of the mortality probabilities. Using this revised projected mortality table instead of the previous version (of 2005) leads to an increase in life expectancy, and hence, it leads to an increase in the value of pension fund's liabilities. This further reduces the funding ratio of pension funds. An example of the differences between the projected mortality table in 2005 and 2010 is given in Table 1.2 which displays the (remaining) life expectancy in 2050 of both males and females at birth and at age 65.

Table 1.2: **Life expectancy in 2050 for the Netherlands**, using projected life tables from the AG.

	model 2005	model 2010	change
0 year-old male	82.5	85.5	+3.0
0 year-old female	84.3	87.3	+3.0
65 year-old male	19.6	22.0	+2.4
65 year-old female	21.3	23.8	+2.5

The table displays the period life expectancy in 2050, using projected life tables from the actuarial society (AG) in 2005 ("model 2005") and in 2010 ("model 2010") in the Netherlands. The period life expectancy in 2050 is calculated using the projected survival probabilities in the year 2050. Source: Actuarial society, http://www.ag-ai.nl/.

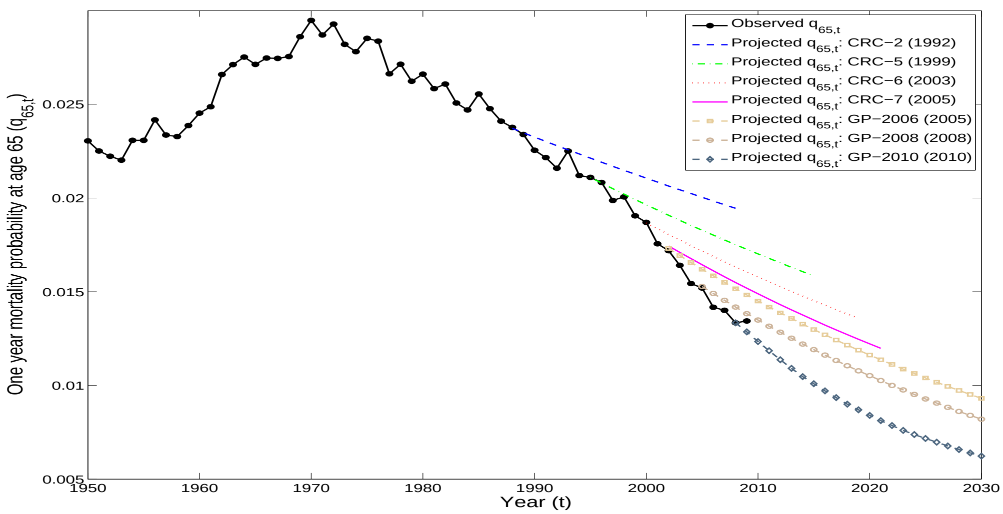
The difference in life expectancy between the two succeeding projected life tables determined by the actuarial society is large. The increase in projected life expectancy is partly due to a change in the method to determine the future survival probabilities and partly due to a larger than expected increase in realized survival probabilities between the two forecasts. In the UK, where the actuarial society projected the life expectancy from 1971 onwards, the realization of the life expectancy at birth is typically higher than the projected ones. This indicates that in the UK the increase in survival probabilities have been consistently underestimated. Unfortunately, the

[4]Source: Dutch Central Bank (DNB), Table T8.8. These numbers only include pension funds with liabilities, excluding funds which fully reinsured their liabilities. In September 2010 the total number of pension funds in the Netherlands was 602.

actuarial society only projected life tables in 2005 and 2010, so we have too few data to determine whether the underestimation in 2005 is due to the stochasticity in the evolution of the survival probabilities or whether it is due to a consistent underestimation of the increase in future survival probabilities.

In the Netherlands the life tables are also projected by Statistics Netherlands and by the Dutch Association of Insurers. Forecasts of the mortality probabilities were provided by the Group Reference Rate Commission, which in 2005 became the Commissie Pensioen- en Lijfrentetafels, a commission of the Dutch Association of Insurers, of Dutch males aged 65 are displayed in Figure 1.1. Statistics Netherlands provided five consecutive projected life tables. Table 1.3 displays the life expectancy in 2049 using projected life tables from Statistic Netherlands.

Figure 1.1: **Observed and forecasted mortality probability** for a male in the Netherlands aged 65



The figure displays the observed and forecasted one year mortality probability of Dutch males aged 65. The mortality probabilities are provided by the Group Reference Rate Commission (in Dutch: "Commissie Referentietarief Collectief", CRC) and PLT (Commissie Pensioen- en Lijfrentetafels). The projected tables are "CRC-2" published in 1992, "CRC-5" published in 1999, "CRC-6" published in 2003, "CRC-7" published in 2005, "GP-2006" published in 2005, "GP-2008" published in 2008, "GP-2010" published in 2010.

Table 1.3: **Life expectancy in the Netherlands** using projected life tables from Statistics Netherlands.

	CBS 2000-2049	CBS 2002-2049	CBS 2004-2050	CBS 2006-2050	CBS 2008-2050
0 year-old male	79.51	79.51	79.52	81.43	83.08
0 year-old female	82.60	82.51	82.61	84.13	85.47
65 year-old male	17.96	18.08	17.86	19.20	20.56
65 year-old female	20.56	20.50	20.44	21.52	22.63

The table displays the period life expectancy in 2049, using projected life tables from Statistic Netherlands in 2000 ("CBS 2000-2049"), 2002 ("CBS 2002-2049"), 2004 ("CBS 2004-2050"), 2006 ("CBS 2006-2050"), and 2008 ("CBS 2008-2050"). The period life expectancy in 2049 is calculated using the projected survival probabilities in the year 2049. Source: Statistics Netherlands, http://statline.cbs.nl/.

From Figure 1.1 and Table 1.3 we observe that the uncertainty in the projected mortality tables is significant. For pension funds and insurer this poses a problem for the valuation of life insurance products. The uncertainty in future mortality probabilities has significant economic consequence for insurers and pension funds. This thesis investigates various aspects of longevity risk and in the remainder of this chapter we provide an overview of the chapters in this thesis.

Chapter 2 ("*Longevity risk*") reviews the current state of the literature concerning longevity risk. First, we discuss the modeling of future mortality, including the Lee and Carter (1992)-approach, as well as other approaches. Second, we discuss the importance of longevity risk for the solvency of portfolios of pension and life insurance products. Finally, we investigate possibilities for longevity risk management. In particular, we consider longevity risk management through securitization and/or pension and insurance (re)design.

In order to have a good risk management for life insurers, who may be faced with significant longevity risk, a good valuation of life insurance products is necessary. In **Chapter 3** ("*Calculating Capital Requirements*") we quantify the valuation and the reserve requirements for life insurance products using the cost of capital method. The goal of the chapter is twofold. First, using an internal model which is in line with the Solvency II proposal for life insurers, we derive a closed form approximation for the capital requirements for different portfolios of life insurance products, in case mortality rates are forecasted based on the Lee and Carter (1992) model. The number of simulation required to be sufficiently accurate would be exponential in the number of years to maturity of the life insurance contract. The approximated distribution allows us to calculate solvency requirements with good accuracy, but without simulations for several life insurance products. Using a market-to-model model, we calculate the market value of the liabilities and the capital reserve that is needed in order to limit the probability of shortfall within a year to 0.5%. Second, using the internal model we quantify the effects of different simplifications made in the Solvency II proposal on the capital requirements.

Given a good quantification of longevity risk, a pension fund or insurer may try to reduce the risk in its portfolio. In this thesis we propose two methods, namely through pension plan design and hedge effects of combining different life insurance products. In **Chapter 4** ("*The effect of product design*") we investigate the effect of a defined benefit pension plan design on the longevity risk of a portfolio of life insurance products. We consider defined benefit pension plans that, at retirement age, allow the participant to choose between a single life annuity and a joint and survivor annuity. We compare two plans that differ in terms of how pension rights are accrued. In one plan, the participant accrues the right to receive a single life annuity, and can exchange that annuity for an actuarially equivalent joint and survivor annuity at retirement date. The opposite holds in the other plan. We show that both plans are affected by longevity risk in two ways. First, the participants' choices at retirement age affect the ratio of survivor benefits over single life benefits, and, therefore, affect the natural hedge potential that arises from combining single life and survivor annuities. Second, uncertainty in the rate at which the participant will be allowed to exchange one type of annuity for the other at retirement date induces uncertainty in the level of the nominal rights for single life and survivor annuities, respectively. We compare the two plans, and show that longevity risk is substantially lower in case rights are accrued in the form of a joint and survivor annuity.

In **Chapter 5** ("*Hedge effects in a portfolio of life insurance products with investment risk*") we investigate the hedge effect of combining different life insurance products on longevity risk of a portfolio of life insurance products. Existing literature shows that the effect of longevity risk on single life annuities can be substantial, and that there exists a (natural) hedge potential from combining single life annuities with death benefits or from investing in survivor swaps. The effect of unhedgeable financial risk on these hedge effects is typically ignored. The aim of this chapter is to quantify longevity risk in portfolios of mortality-linked assets and liabilities, taking into account the effect of unhedgeable financial risk. We find that unhedgeable investment risk significantly affects the impact of longevity risk in life insurance products. It also significantly affects the hedge potential that arises from combining life insurance products, or from investing in longevity-linked assets. For example, our results suggest that ignoring the effect of unhedgeable financial risk can lead to severe overestimation of the natural hedge potential from death benefits, and underestimation of the hedge effects of survivor swaps.

Finally, we consider the effect of longevity risk on the investment and consumption decisions for individuals, in particular we focus on the optimal annuitization decision. In **Chapter 6** ("*Annuity decisions with systematic longevity risk*") we investigate the effect of systematic longevity risk on the attractiveness of different types of annuities. We consider a life-cycle framework with expected utility where an individual faces both investment and longevity risk. In contrast to existing literature we allow not only for idiosyncratic, but also for systematic longevity risk. When comparing the expected lifetime utility, conditional on the type of annuity which is purchased, we find for a 65-year old male that (i) systematic longevity risk reduces the attractiveness of annuities, (ii) when an immediate annuity is purchased, the expected lifetime utility is decreasing in the postponement period, (iii) when in the future purchasing an immediate annuities, the effect of the evolution of the survival probabilities on the optimal fraction of annuitized wealth is large, and (iv) the optimal annuity to purchase at retirement is a deferred annuity which starts to pay after only a short deferral period. However, when the purchase of an annuity with the optimal deferral period is compared to the purchase of an immediate annuity at retirement date, the utility gain is negligibly small.

This thesis provides findings for quantifying and managing longevity risk in portfolios of life insurance products for life insurers, pension funds, and individuals. Quantifying longevity risk, which can be done using different methods as described in this thesis, is the first step. The next step is to reduce longevity risk by the plan design and by capital market solutions. In the decision whether to hedge longevity risk or not, both the price and the hedge effectiveness of the products will play an important role. The price of longevity risk will also play a role for individuals, since retirement products, such as a single life annuity, will include a risk margin for longevity risk in their price. This implies that longevity risk also impacts the decisions of individuals. To sum up, with the help of new (innovative) financial products to hedge longevity risk and findings discussed in this thesis, pension funds, life insurers, and individuals should be aware of their longevity risk and manage it optimally.

Chapter 2

Longevity Risk

This chapter is based on De Waegenaere, Melenberg and Stevens (2010).

2.1 Introduction

Most of the western world has seen a steady increase in the average lifetime of its inhabitants over the past century. For example, the expected remaining lifetime of a Dutch male aged 65 increased from 13.5 years in 1975 to almost 17.5 years in 2009.[1] The potential effects of trends in mortality on pension costs present significant challenges for governments as well as individual pension funds and life insurers. Biffis and Blake (2009) report that every additional year of life expectancy at age 65 is estimated to add at least 3% to the present value of UK pension liabilities and calculations from Eurostat indicate that an increase in the life expectancy at birth of one year leads to an increase of, an average, 0.3% of the GDP of the public pension expenditure in the EU, see European Commission and the Economic Policy Committee (2009). This clearly illustrates the need to consider interventions that can mitigate the adverse effects on pension and insurance providers, while still guaranteeing an adequate level of retirement and insurance benefits to policyholders. Identifying appropriate interventions is challenging. The major challenge, however, is not in the trend itself, but in the fact that the future development of life expectancy is uncertain. Indeed, although the past trends suggest that further changes in mortality rates are to be expected, there is considerable uncertainty regarding the future development of mortality. Decisions regarding redesign of pension and insurance

[1] Source: Statistics Netherlands, http://statline.cbs.nl/.

systems should therefore appropriately account for the effects of this particular uncertainty on the costs of pensions. In addition, since interventions in the design of pension and insurance contracts can mitigate, but not eliminate, the effects of mortality risk, there will be residual risk. Whereas the focus of regulators has long been on the risk in *financial investments*, there is now increasing awareness that accurate quantification and management of the risk in pension and insurance *liabilities* is equally important. For example, the Solvency II project (Group Consultatif Actuariel Europeen, 2006), the goal of which is to redesign financial regulation of insurance companies in the EU, has put increased emphasis on the valuation and management of pension and insurance liabilities. Common approaches taken in practice to deal with the effect of changes in life expectancy have included regularly re-estimating the value of the liabilities on the basis of newly estimated death probabilities, or determining the value of the liabilities on the basis of a projected trend in mortality. These approaches, however, are either retrospective, or do not properly account for the uncertainty in the future development of mortality. Risk management practices may need to be adjusted in order to account properly for uncertainty in the future development of mortality.

This chapter reviews the literature on longevity risk (i.e., the uncertainty in future changes in mortality rates). The focus is on models to forecast the probability distribution of future mortality rates, approaches to quantify the effect of longevity risk on pension and insurance liabilities, and possibilities for risk management.

The chapter is organized as follows. In the next section, we formally define longevity risk, and discuss the distinction to individual mortality risk. We also show that, in contrast to individual mortality risk, longevity risk does not become negligible when portfolio size becomes large. Next, in Section 2.3 we review the literature on mortality modeling, including the Lee and Carter-approach, which is nowadays used extensively to model the uncertainty in the probability distribution of future mortality. In addition to the original Lee and Carter (1992)-model, we discuss several alternative approaches. Moreover, we decompose longevity risk into process risk and model risk, where the latter includes as special case parameter risk. Model risk arises due to a lack of knowledge regarding the correct probability distribution of future mortality rates, and process risk is the uncertainty in the mortality trends that remains, even in case we exactly would know the correct probability distribution of future mortality rates. Parameter risk is model risk that arises due to sampling inaccuracy, given a selected model (class), like the Lee and Carter-model.

In Section 2.4 we discuss approaches to quantify the importance of longevity risk for portfolios of (pension) annuities. First, we use the *discounted present value of liabilities* to demonstrate the relative importance of individual mortality risk and longevity risk, and the effect of portfolio size, which is an extension of Olivieri (2001). Second, we briefly discuss the *funding ratio* approach to determine longevity risk in portfolio of life insurance products. Third, we discuss the *probability of ruin* approach, i.e., quantifying longevity risk by the capital required to reduce the probability that, for a given (re)investment strategy, the current value of the assets will not be sufficient to meet all future liabilities to a certain percentage.[2]

Finally, in Section 2.5 we investigate possibilities for longevity risk management for life insurers and pension funds, following Cairns et al. (2008a). We illustrate some aspects of longevity risk management, in particular, the determination of solvency buffers, and the effect of the product mix as a natural approach to diversify longevity risk. We also briefly discuss the attempts to set up a "life market," a trading place for mortality-based products, that could be used to hedge or to reduce the longevity risk. Section 2.6 concludes.

2.2 Longevity Risk

In this section, we first demonstrate the importance of longevity trends for annuity providers. Then, we discuss the distinction between longevity risk and mortality risk, and provide evidence that longevity risk is substantial. Finally, we discuss the implications of longevity risks for pricing annuities (or other longevity related assets and liabilities), as well as for risk management practices.

2.2.1 Mortality Trends

We first introduce some basic terminology and results related to mortality. For the sake of argument, first assume that future death probabilities are known with certainty. An important quantity is the "one-year death probability," denoted by $q_{x,t}^{(g)}$, which quantifies the probability that a person of age x at year t and belonging to group g will not survive another year. The probability that the same individual

[2] We would like to emphasize that these studies not only use different approaches to quantify longevity risk, but also use different models to forecast future mortality. Any difference in the magnitude of longevity risk between these studies can be due to either the choice of method or the choice of forecast model.

survives at least another year is then given by

$$\begin{aligned} p_{x,t}^{(g)} =& \mathbb{P}\left(T_{x,t} \geq 1 | T_{x,t} \geq 0\right) \\ =& 1 - q_{x,t}^{(g)}, \end{aligned} \tag{2.1}$$

where $T_{x,t}$ denotes the random remaining lifetime of an individual aged x at time t. If, for example, group g (Dutch males or Dutch females) is understood, we suppress the superindex (g). Moreover, if the probabilities would be independent of time t, we can simplify even further, by writing q_x and p_x. Assuming this for the moment, the probability that the same individual (of age x and belonging to group g, suppressed) survives at least τ more years is then given by

$$\begin{aligned} {}_\tau p_x =& \mathbb{P}\left(T_{x,t} \geq \tau\right) \\ =& \prod_{s=1}^{\tau} \mathbb{P}\left(T_{x,t} \geq s | T_{x,t} \geq s-1\right) \\ =& \prod_{j=0}^{\tau-1} p_{x+j}, \end{aligned} \tag{2.2}$$

where ${}_1p_x = p_x$. Using these probabilities, we can derive e_x, the expected number of years the individual will survive:

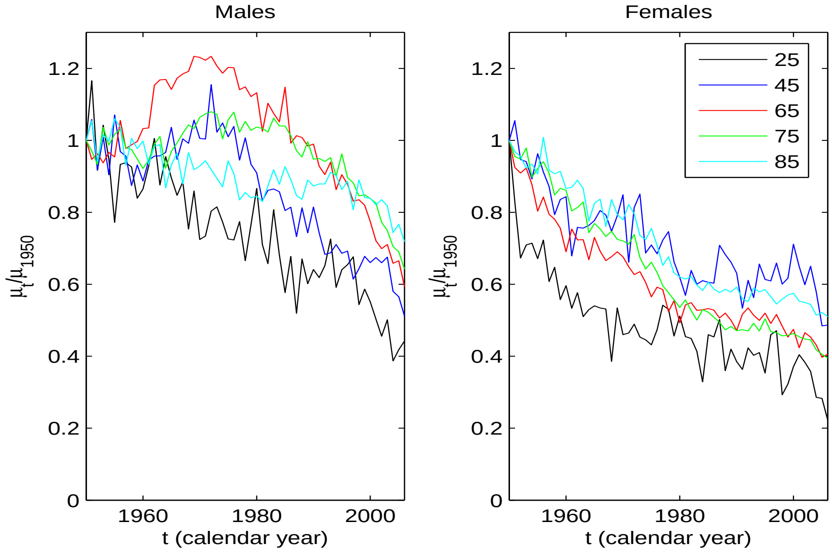
$$e_x = \sum_{\tau \geq 1} {}_\tau p_x. \tag{2.3}$$

Thus, seen from year t this individual is expected to die in year $t+e_x$, at age $x+e_x$.

The above, however, assumes that one-year death probabilities are *constant* over time. There is ample evidence that death probabilities change over time. In Figure 2.2.1 we plot the one-year death probability $q_{x,t}^{(g)}$ for a number of different ages x and two groups g, namely the group of Dutch males and the group of Dutch females, for the years $t = 1950$ to $t = 2006$, where we normalize by the one-year death probabilities of year $t = 1950$. These one-year death probabilities are obtained from the Human Mortality Database.[3]

[3]See www.mortality.org.

Figure 2.2.1: **One-year death probabilities**



This figure plots the observed one-year death probabilities for Dutch males (left panel) and Dutch females (right panel), for different ages and for different time periods, normalized to one for the year 1950. The data originates from the Human Mortality Database.

This figure clearly illustrates that, at least over longer periods, the one-year death probabilities decrease over time, reflecting the increase in longevity over time. But then the assumption that the one-year death probabilities are *constant* over time is not valid. Following Cairns, Blake and Dowd (2006), we define the *realized* one-year survival probabilities of the cohort aged x at date t as follows:

$$p_{x+s,t+s} = 1 - q_{x+s,t+s} = \mathbb{P}\left(T_{x,t} \geqslant s+1 | T_{x,t} \geqslant s, \mathcal{F}_\infty\right), \tag{2.4}$$

where $q_{x+s,t+s}$ is the realized one-year death probability in year $t+s$ of the cohort aged x at date t, $\mathcal{F}_\infty$ denotes the set that contains all information regarding mortality rates at all (future) dates, and where $T_{x,t}$ denotes the random remaining lifetime of an individual aged x in year t. Moreover, we denote

$$_\tau p_{x,t} = \mathbb{P}\left(T_{x,t} \geqslant \tau | \mathcal{F}_\infty\right).$$

Then,

$$\begin{aligned}
{}_\tau p_{x,t} &= \mathbb{P}\left(T_{x,t} \geqslant \tau | T_{x,t} \geqslant \tau - 1, \mathcal{F}_\infty\right) \cdot \mathbb{P}\left(T_{x,t} \geqslant \tau - 1 | \mathcal{F}_\infty\right) \\
&= \prod_{s=1}^{\tau} \mathbb{P}\left(T_{x,t} \geqslant s | T_{x,t} \geqslant s - 1, \mathcal{F}_\infty\right) \\
&= \prod_{s=0}^{\tau-1} p_{x+s,t+s}. \qquad (2.5)
\end{aligned}$$

For the models used in this thesis, it holds true that conditional on given death rates up to time $t + s$, whether the individual survives until time $t + s$ is independent of death rates in years beyond that date. Therefore, ${}_\tau p_{x,t}$ is realized at date $t + \tau$, and is a random variable prior to that date (see, e.g., Cairns, Blake and Dowd, 2006).

Conditional on $\mathcal{F}_\infty$, the expected number of years that a person of age x at year t will survive is given by

$$e^{(g)}_{x,t} = \sum_{\tau \geq 1} {}_\tau p^{(g)}_{x,t}, \qquad (2.6)$$

instead of (2.3). Thus, to calculate (2.6), we need *future projections* of the one-year death probabilities $q^{(g)}_{x,t'}$, for $t' \geq t$. Let us first consider deterministic projections. Such projected one-year death probabilities might result in a serious underestimation of the expected number of years an individual will survive and of the expected discounted value of the annuity. Indeed, the projected life table from the actuarial society[4] show that the expected remaining lifetime changes substantially when future changes in mortality rates are taken into account. For the age $x = 65$, they report an (period) expected remaining lifetime for males of 17.4 years and for females 19.2 years in 2011 using equation (2.3) and an (cohort) expected remaining lifetime in 2011 for males of 20.6 years and for females of 22.10 years when using using equation (2.6).

Such trends obviously have important implications for the value of pension annuities. As reported in, for instance, Biffis and Blake (2009), every additional year of life expectancy at age 65 is estimated to add at least 3% to the present value of UK pension liabilities. Assuming that such numbers apply more generally, the economic implications of longevity become obvious.

This is confirmed by results from Hári et al. (2008b), who illustrate the effect of longevity trends on the expected present value of annuity payments. Specifically, they consider a (deferred) annuity that pays off one Euro (in arrears) every year

[4]Source: projected life table in 2010 by the actuarial society, http://www.ag-ai.nl/.

that the annuitant survives, and is older than 65. Conditional on $\mathcal{F}_\infty$, the expected present value, at time t, for an annuitant aged x belonging to group g is given by:

$$\widetilde{a}_{x,t}^{(g)} = \sum_{\tau \geq \max\{65-x,0\}} {}_\tau p_{x,t}^{(g)} \cdot P_t^{(\tau)}, \tag{2.7}$$

where $P_t^{(\tau)}$ denotes the market value, at time t, of a zero-coupon bond maturing at time $t+\tau$ (i.e., the date-t value of one Euro to be paid in period $t+\tau$). They find that the present value of annuity payments based on period life tables underestimates[5] the value based on forecasted death probabilities by 7.7% for a 25-year-old man and 8.8% for a 25-year-old woman. For the 65-year-old, the corresponding numbers are 0.4% and 1.7%, respectively.

2.2.2 Sources of Mortality Risk

While the above illustrates the importance of mortality trends for pension providers, there is at hand a more challenging issue. Indeed, Figure 2.2.1 shows not only that the one-year death probabilities (on average) decrease over time, but also that this decrease is different for various ages and different for males and for females in an (at least to some extent) *unpredictable* way. When extrapolating this finding to forecasting future one-year death probabilities, it seems quite implausible to assume that we would be able to know these future one-year death probabilities in a *deterministic* way, without any uncertainty. Instead, it would seem more realistic to deal with this uncertainty, by assuming that the one-year death probabilities $q_{x,t'}^{(g)}$ are *stochastic* at time t, for $t' > t$. If so, we are confronted with *longevity risk*: the probability at year t that an individual of age x and belonging to group g survives at least τ other years (see (2.5)) is not known deterministically, but is random. The literature therefore distinguishes two sources of mortality risk:[6]

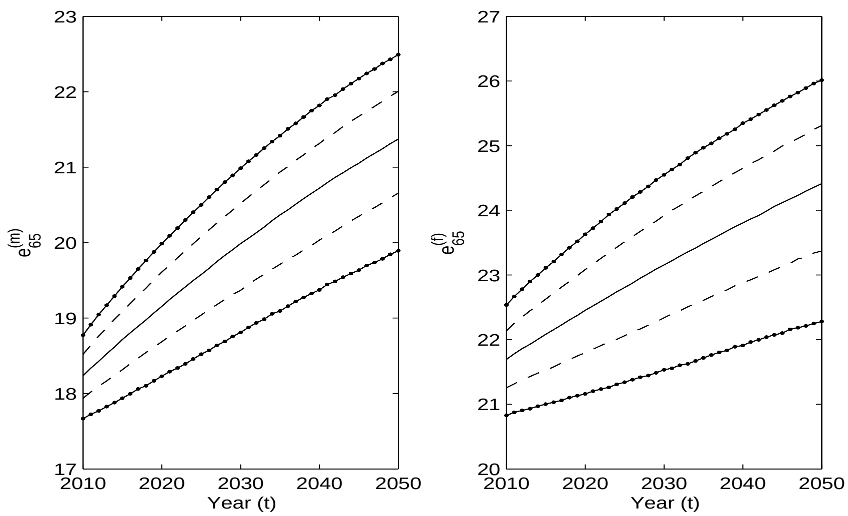
- *Individual mortality risk* refers to the risk due to the fact that, for given death probabilities, an individual's remaining lifetime is a random variable;
- *Longevity risk* refers to the risk as a consequence of *longer term* deviations from deterministic mortality projections.

[5] There are some exceptions for elderly men, due to the specific forecasted mortality rates employed by Hári et al. (2008b).

[6] The literature also distinguishes so-called *mortality catastrophe risk*, which relates to the risk of higher than expected mortality (for example due to an epidemic). The focus in this chapter is on individual mortality risk, and, more importantly, longevity risk.

As a consequence of longevity risk, the expectation of the number of years that a person of age x at date t will survive (as well as all other quantities that depend on future one-year death probabilities), conditional on $\mathcal{F}_\infty$, becomes a random variable at date t (see (2.6)).

Figure 2.2.2: **Expected Remaining Lifetimes**



In this figure we plot quantiles (10%, 25%, 50%, 75%, 90%) of the distribution of the expected remaining lifetime $e_{x,t}^{(g)}$ for the group g of Dutch males (left panel) and Dutch females (right panel) of age $x = 65$, for the years $t = 2007$ to 2050. The quantification of the longevity risk is described in the Appendix of Chapter 5.

Figure 2.2.2 illustrates the evolution of the probability distribution of the expected remaining lifetimes $e_{x,t}^{(g)}$ for the groups g of Dutch males and Dutch females of age $x = 65$, for the years $t = 2010$ to 2050, when the future death probabilities are assumed to be random, as will be described in the next section.[7] The graph shows a number of quantiles (ranging from the 0.10- to the 0.90-quantile). The figure shows that there is already substantial longevity risk in the earliest projections corresponding to $t = 2010$.[8] The quantile intervals for the remaining lifetime of a

[7]We use the quantification described in the Appendix of Chapter 5, which allows for both process and model risk, to be explained in the next section.

[8]It is a picture similar to that in Dowd, Blake, and Cairns (2008), see also Biffis and Blake (2009), who consider the UK population. The figure clearly illustrates that the expected remaining lifetimes of 65-years old are projected to increase in the future.

65-year old in $t = 2050$ are even much wider.

The significant degree of uncertainty in future expected lifetimes suggests that the effect of uncertain changes in mortality on the value of pension liabilities may also be substantial.

2.2.3 On the Importance of Longevity Risk

In this section, we demonstrate that longevity risk, in contrast to individual mortality risk, can not be diversified away by increasing portfolio size. We discuss the implications for the pricing of longevity-linked assets or liabilities, as well as for the risk management practices of pension funds.

In order to do so, we consider a pool of immediate life annuities sold to N individuals of age x belonging to group g in year t. The annuity pays off one Euro (in arrears) to an individual every year that this individual survives. Assume a constant and risk free annual interest rate r, and denote by $1_{i,t+\tau}$ the dummy variable equal to one in case annuitant i is still alive at time $t+\tau$. Then the present value at time t of the annuity payments to annuitant i in years $t+\tau$, $\tau \geqslant 1$, is given by

$$Y_i = \sum_{\tau \geqslant 1} 1_{i,t+\tau} \frac{1}{(1+r)^{\tau}}. \tag{2.8}$$

For the sake of argument, first assume that future death probabilities are known with certainty (i.e., there is individual mortality risk, but no longevity risk). Then, the expected present value at time t of the annuity payments to annuitant i is given by

$$a_{x,t}^{(g)} = \sum_{\tau \geq 1} \mathbb{E}\left[1_{i,t+\tau}\right] \frac{1}{(1+r)^{\tau}} = \sum_{\tau \geq 1} {}_{\tau}p_{x,t}^{(g)} \frac{1}{(1+r)^{\tau}}. \tag{2.9}$$

Then, using a pooling argument, this expected discounted value is also the fair price of the annuity. The fair price of Y_i will be equal to the fair price of $\frac{1}{N}\sum_{i=1}^{N} Y_i$. Assume that the Y_i are independent, with expected value $\mu = \mathbb{E}(Y_i)$ and variance $\sigma^2 = \text{Var}(Y_i)$. Then the variance of $\frac{1}{N}\sum_{i=1}^{N} Y_i$ can be calculated as

$$\text{Var}\left(\frac{1}{N}\sum_{i=1}^{N} Y_i\right) = \sigma^2/N. \tag{2.10}$$

In case N becomes very large, $\frac{1}{N}\sum_{i=1}^{N} Y_i$ becomes risk free, and its fair price (like the fair price of Y_i) equals its expected discounted value, i.e., there is no risk premium.[9] Thus, the one-year death probabilities $q_{x,t}^{(g)}$, and the corresponding survival probabilities as defined in (2.1) and (2.2), represent mortality risk at the individual level, which, however, can be eliminated by an insurance company or pension fund by means of pooling. As a consequence, this individual mortality risk should not be priced.

With longevity risk, however, the fair price of the annuity (and other products with a payoff that depends on future survival outcomes) typically will include a (longevity) risk premium. To illustrate this, we return to the annuity portfolio (see (2.8)). Conditional upon the future death rates at time t, given by the set $\mathcal{F}_\infty$ it still makes sense to assume that the payoffs Y_i are independent, with now mean $\mu(\mathcal{F}_\infty)$ and variance $\sigma^2(\mathcal{F}_\infty)$, both depending on $\mathcal{F}_\infty$. However, when calculating the (unconditional) variance of $\frac{1}{N}\sum_{i=1}^{N} Y_i$ we have to take into account that conditional on $\mathcal{F}_\infty$, the one-year probabilities, are random due to longevity risk. We find

$$\mathrm{Var}\left(\frac{1}{N}\sum_{i=1}^{N} Y_i\right) = \mathbb{E}\left(\mathrm{Var}\left(\frac{1}{N}\sum_{i=1}^{N} Y_i\right) \middle| \mathcal{F}_\infty\right) + \mathrm{Var}\left(\mathbb{E}\left(\frac{1}{N}\sum_{i=1}^{N} Y_i\right) \middle| \mathcal{F}_\infty\right).$$

Therefore,

$$\mathrm{Var}\left(\frac{1}{N}\sum_{i=1}^{N} Y_i\right) = \mathbb{E}\left(\sigma^2\left(\mathcal{F}_\infty\right)\right)/N + \mathrm{Var}\left(\mu\left(\mathcal{F}_\infty\right)\right). \tag{2.11}$$

The first term on the right hand side (corresponding to the pooling effect) still vanishes with increasing N. However, the second term (reflecting the effect of longevity risk) is independent of N. Thus, with longevity risk, even when N becomes very large, $\frac{1}{N}\sum_{i=1}^{N} Y_i$ does not become risk free anymore. As a consequence, the pooling argument no longer results in an elimination of mortality risk: longevity risk remains, and products whose payoffs depend on future mortality typically will include

[9] A no arbitrage argument goes as follows. Let $Y_{i,\tau}$ denote the payoff of the annuity at time $\tau = t' - t$, and let p denote the no arbitrage price of $Y_{i,\tau}$. Suppose M_τ is the relevant Stochastic Discount Factor, such that $p = \mathbb{E}(M_\tau Y_{i,\tau})$. Then, assuming that the $M_\tau Y_{i,\tau}$ are identically distributed for different i, we have

$$p = (M_\tau Y_{i,\tau}) = \mathbb{E}\left(M_\tau\left(\frac{1}{N}\sum_i Y_{i,\tau}\right)\right) = \mathbb{E}(M_\tau)\,\mathbb{E}\left(\frac{1}{N}\sum_i Y_{i,\tau}\right) + \mathrm{Cov}\left(M_\tau, \frac{1}{N}\sum_i Y_{i,\tau}\right).$$

Using $\left|\mathrm{Cov}\left(M_\tau, \frac{1}{N}\sum_i Y_{i,\tau}\right)\right| \leq \sigma(M_\tau)\,\sigma(Y_{i,\tau})/N$, we find, for $N \to \infty$, $p = \mathbb{E}(Y_{i,\tau})\,\mathbb{E}(M_\tau) = \mathbb{E}(Y_{i,\tau})\frac{1}{(1+r)^\tau}$.

a (longevity) risk premium. Thus, the expected value $\mathbb{E}(Y_i) = \mathbb{E}(\mu(\mathcal{F}_\infty))$ may no longer be the fair value of the annuity. We shall refer to this expectation as the *best estimate*. From the point of view of an insurer or pension fund, this best estimate might be seen as a lower bound of the value of the annuity (as a liability).

The result that longevity risk cannot be diversified away using pooling has important implications for both pricing and risk management. First, this non-diversifiability implies that the price of a longevity linked asset or liability is likely to include a (longevity) risk premium. However, annuity payoffs (as well as the payoffs of other products depending on future survival outcomes) typically cannot be hedged by currently traded financial products.[10] As a consequence of this market incompleteness, arbitrage arguments are insufficient to obtain unique market prices of annuities and related products. This seriously complicates the fair valuation of liabilities depending on future survival outcomes due to the presence of a (longevity) risk premium. The current literature devotes considerable attention to this pricing problem. Specifically, traditional finance approaches (risk-neutral pricing theories, see, for example, Cairns et al., 2006a) as well as actuarial pricing approaches (Wang's premium principle, see, for example, Lin and Cox, 2005) receive considerable attention. However, market incompleteness implies that calibrating these pricing models remains difficult. For a recent and thorough overview of the literature on pricing longevity risk, we refer to Bauer, Boerger, and Russ (2009).

Second, non-diversifiability has important implications for risk management. Indeed, the traditional approach used in case of individual mortality risk is to reduce the risk by increasing portfolio size, for example, by mutual reinsurance. As discussed above, however, increasing the portfolio size does not reduce the impact of longevity risk, so that other risk management tools need to be applied. In order to investigate this further, a first important step is the modeling of the probability distribution of future mortality, which we will discuss in the next section. In Section 2.4, we then illustrate how mortality models can be used to quantify the effect of longevity risk, and evaluate the effectiveness of risk management practices in the presence of longevity risk.

[10] Sometimes, there are natural hedge possibilities, see, for example, Milevsky and Promislow (2001) or Cox and Lin (2007). See also Chapter 5.

2.3 Modeling Future Mortality

In this section we discuss the quantification of the uncertainty in the probability distribution of future mortality. Reviews of such a quantification include Booth (2006), Pitacco (2004), Tabeau (2001), and the recent monographs by Girosi and King (2008) and Pitacco, Denuit, Haberman, and Olivieri (2009). See also Benjamin and Soliman (1993), Delwarde and Denuit (2006), Cairns, Blake, and Dowd (2008a) and Hári (2007).

The starting point of the analysis is the *(raw) central death rate*[11] or observed per capita number of deaths, defined by $m_{x,t}^{(g)} = D_{x,t}^{(g)}/E_{x,t}^{(g)}$, where $D_{x,t}^{(g)}$ denotes the number of people with age x in group g that died in year t, and where $E_{x,t}^{(g)}$ denotes the so-called exposure, being the number of person years in group g with age x in year t. These central death rates are typically observed on a yearly basis, ranging from age $x = 0$ to some maximum age, like $x = 110$, while the time index t ranges from some starting year, normalized as $t = 1$ up to some recent year $t = T$. The number of deaths $D_{x,t}^{(g)}$ and the exposure $E_{x,t}^{(g)}$ can be obtained from population statistics, where the exposure is usually approximated.[12] Given $m_{x,t}^{(g)}$ for all age groups x, one can calculate the one-year death probabilities $q_{x,t}^{(g)}$, see, for example, McCutcheon and Nesbitt (1973). However, since this is a complicated relationship, one usually makes some additional assumptions to obtain an easier link between $m_{x,t}^{(g)}$ and $q_{x,t}^{(g)}$. For instance, assuming that the exposure is linear in x, results in the relationship

$$q_{x,t}^{(g)} = \frac{m_{x,t}^{(g)}}{1 + \frac{1}{2} m_{x,t}^{(g)}}. \tag{2.12}$$

Alternatively, one makes assumptions such that the central death rate equals the so-called force of mortality,[13] in which case one obtains

$$q_{x,t}^{(g)} = 1 - \exp\left(-m_{x,t}^{(g)}\right). \tag{2.13}$$

When quantifying longevity risk, one typically models the evolution of the raw central death rate $m_{x,t}^{(g)}$ or the one year death probabilities $q_{x,t}^{(g)}$ over time for a given

[11] Raw refers to as observed in the data.

[12] For more details, see, for instance, Gerber (1997) or the technical report corresponding to the Human Mortality Database.

[13] The force of mortality is defined as $\mu_{x,t}^{(g)} = \lim_{\Delta t \to 0} P\left(0 \le T_{x,t}^{(g)} \le \Delta t\right)/\Delta t$, where $T_{x,t}^{(g)}$ denotes the remaining lifetime at time t of an individual of age x belonging to group g. When the force of mortality is constant within bands of time, i.e., $\mu_{x,t+\tau} = \mu_{x,t}$, for $0 \le \tau < 0$, then the force of mortality equals the central death rate. See, for instance, Gerber (1997) for further details.

group g. In case of the central death rate this results in a decomposition of the raw central death rate in a systematic part, say $\widetilde{m}_{x,t}^{(g)}$, and a remaining idiosyncratic part. The systematic part is then projected into the future, and equations (2.12) or (2.13) are used to find the projected future one-year death probabilities, using the systematic part of the central death rates, instead of the raw versions. In case of the one year death probabilities the modeling will result in a decomposition a systematic and idiosyncratic part, but now in terms of these one year death probabilities, and again the systematic part ($\widetilde{q}_{x,t}^{(g)}$) is projected into the future. Since the models used to quantify central death rates or the one year death probabilities typically consider a fixed group g, we shall suppress the superindex g in the remainder of this section. In the next section, we first briefly review the earlier modeling of mortality. In Section 2.3.2 we review the Lee and Carter (1992)-approach, while in Section 2.3.3 we discuss some recent developments.

2.3.1 Dynamic Mortality Laws

For a given time period t, $\widetilde{m}_{x,t}$ or $\widetilde{q}_{x,t}$ might be parameterized in some particular way. Such parameterizations are often called "mortality laws," describing mortality (at time t) as a function of age x. Early mortality laws include the "Gompertz law" (Gompertz, 1825), "Makeham's law"(Makeham, 1860), and "Thiele's law" (Thiele, 1872). A more recent version is the "Heligman-Pollard law" (Heligman and Pollard, 1980), which states (for some given time t, with t suppressed)

$$\widetilde{q}_x = A^{(x+B)^C} + D\exp\left(-E\left(\log x - \log F\right)^2\right) + \frac{GH^x}{1+GH^x}, \tag{2.14}$$

where A-H are (unknown) parameters. This law consists of three components, the first of which aims to capture infant and childhood mortality, the second one adult mortality,[14] and the third one the mortality of the elderly.

An obvious way to obtain dynamic mortality models, is to fit some given mortality law each period t for which data is available, with some or all parameters time dependent. The resulting time series of time-dependent parameter values can then be quantified using appropriate statistical or econometric models. Using such models makes forecasting future mortality trends as well as quantifying longevity risk a straightforward exercise, at least, theoretically. However, as argued by, for instance, Tabeau (2001), fitting mortality laws per period with time dependent parameters,

[14] More precisely, the so-called "accident hump," see Figure 2.3.1.

typically generates rather unstable results, making forecasting mortality trends using this approach from a practical point of view quite difficult, if not impossible. One way to avoid the instability is to combine a mortality law with age and time dependent polynomials, see, for instance, Renshaw, Haberman, and Hatzopoulos (1996). Using polynomials of sufficient order allows quite an accurate in-sample fit. However, using higher order polynomials to make out-of-sample forecasts typically does not work well, see, for example, Bell (1984) for further clarification.

2.3.2 The Lee and Carter Approach

Lee and Carter (1992) propose a parsimonious dynamic mortality model that turned out to perform quite well. The model postulates

$$\ln(m_{x,t}) = \alpha_x + \beta_x \kappa_t + \epsilon_{x,t}, \tag{2.15}$$

with time-independent parameters α_x and β_x, and a (white noise) error term $\epsilon_{x,t}$, where $\{\kappa_t\}$ is a one-dimensional underlying time-dependent latent process that quantifies the evolution of mortality over time. The parameter α_x quantifies the level of the log central death rate of age x, while the parameter β_x quantifies the age x-specific sensitivity of the log central death rate to changes in the group-wide evolution (improvement) as represented by κ_t. The error term $\epsilon_{x,t}$ captures the age and time specific variations around the systematic trend. Due to lack of identification, Lee and Carter (1992) normalize by setting $\sum_x \beta_x = 1$ and $\sum_t \kappa_t = 0$, where the first sum is over all available ages and the second sum over all time periods available in the sample.

Lee and Carter (1992) proposed estimating the model in three steps. In the first step, Singular Value Decomposition (SVD) is applied to find the unique least squares solution (given the normalizations) yielding $\{\widehat{\kappa}_t\}$, $\{\widehat{\alpha}_x\}$, and $\left\{\widehat{\beta}_x\right\}$. The estimated $\{\widehat{\kappa}_t\}$ are then adjusted to ensure equality between the observed and model-implied number of deaths in a certain period (i.e., $\{\widehat{\kappa}_t\}$ is replaced by $\{\widetilde{\kappa}_t\}$) such that

$$\sum_x D_{x,t} = \sum_x \left[E_{x,t} \exp\left(\widehat{\alpha}_x + \widehat{\beta}_x \widetilde{\kappa}_t\right)\right], \tag{2.16}$$

with $D_{x,t}$ the number of deaths and $E_{x,t}$ the exposure, introduced at the beginning of this section. This readjustment is done in order to avoid sizeable differences between the number of observed deaths and the model-implied number of deaths. The systematic part, defined as $\widetilde{m}_{x,t} = \exp(\alpha_x + \beta_x \kappa_t)$, is estimated by $\widehat{\widetilde{m}}_{x,t} =$

$\exp\left(\widehat{\alpha}_x + \widehat{\beta}_x \widetilde{\kappa}_t\right)$. Finally, the Box-Jenkins method is used to identify and estimate the dynamics of the latent factor $\widetilde{\kappa}_t$. Lee and Carter (1992) find as a process for the dynamics of the latent factor a random walk with drift, i.e.,

$$\kappa_t = c + \kappa_{t-1} + \delta_t, \tag{2.17}$$

with c the drift term, and with $\{\delta_t\}$ a white noise process, assumed to follow a normal distribution with mean zero and variance equal to σ_δ^2. The parameters c and σ_δ^2 can be estimated applying standard statistical or econometric time-series techniques.

To avoid the second step of this three-step procedure, Wilmoth (1993) proposed a weighted Singular Value Decomposition. In addition, Lee and Miller (2001) proposed replacing the matching according to equation (2.16) by a matching on the basis of observed and modeled life expectancy. Moreover, these authors suggest restricting the sample period to a recent time period, in order to avoid a potential misspecification due to a violation of the assumption of constant α_x and β_x. Booth, Maindonald, and Smith (2002a) suggest using statistical techniques to select an appropriate sample period, in line with the assumption of constant α_x and β_x.

The Lee and Carter (1992)-model can easily be extended to include more time factors (in addition to κ_t) (see Renshaw and Haberman, 2003a). However, Tuljapurkar, Li, and Boe (2000), investigating the G7 countries (Canada, France, Germany, Italy, Japan, UK, and US),[15] find that a single factor (as in the original Lee and Carter (1992) specification) already suffices to explain over 94% of the variance in the log-specific raw central death rates. Nevertheless, to improve the forecast performance, it might be better to include an additional cohort-specific factor (see Renshaw and Haberman, 2006).

Mortality projections can be obtained by first predicting future values $\widetilde{\kappa}_{T+t}$ (with T the final year of the sample), then predicting the systematic part of the future central death rates as

$$\widehat{\widetilde{m}}_{x,T+t} = \exp\left(\widehat{\alpha}_x + \widehat{\beta}_x \widetilde{\kappa}_{T+t}\right), \tag{2.18}$$

and, finally, calculating the corresponding projected future one-year death probabilities $q_{x,T+t}$, using equation (2.12) or (2.13). Alternatively, Lee and Miller (2001)

[15]For a more recent multi-country comparison of various stochastic mortality models, see, for example, Booth, Hyndman, Tickle, and De Jong (2006).

suggest predicting the future central death rates $\widetilde{m}_{x,T+t}$ using the observed (raw) central death $m_{x,T}$ of the final year in the sample as a jump-off value, i.e., to calculate

$$\widehat{\widetilde{m}}_{x,T+t} = m_{x,T} \exp\left(\widehat{\beta}_x \left(\widetilde{\kappa}_{T+t} - \widetilde{\kappa}_T\right)\right). \tag{2.19}$$

In this way, a jump-off bias can be avoided.

Longevity risk arises, first of all, due to the random character of $\widetilde{\kappa}_{T+t}$, whose exact values are of course unknown at time T, even if its distribution function would be exactly known. This longevity risk is referred to as *process risk*. In addition, there is *model risk*: since we do not know the exact distribution of $\widetilde{\kappa}_{T+t}$, we have to model it, possibly incorrectly, which generates model risk. In particular, if we estimate the probability distribution of $\widetilde{\kappa}_{T+t}$, like in the Lee and Carter-approach, there is model risk due to the sampling error in the estimated parameters $\widehat{\alpha}_x$, $\widehat{\beta}_x$, for all ages x, and in the estimates of the drift term c and variance σ_δ^2 of the random walk. This particular model risk is referred to as *parameter risk*.[16] To quantify these risks, Lee and Carter (1992) suggest using a bootstrap method. They focus on the parameter risk in the time process (2.17) only, arguing that the parameter risk in α_x and β_x is small. Koissi, Shapiro, and Högnäs (2006) extend the bootstrap procedure to include all parameter risk. See also Renshaw and Haberman (2008).

The longevity risk, which consists of process and parameter risk, can be illustrated by a reformulation of the Lee and Carter (1992)-model by Girosi and King (2006). First, let

$$\ell_t = \begin{pmatrix} \ln(m_{1,t}) \\ \vdots \\ \ln(m_{ma,t}) \end{pmatrix}, \tag{2.20}$$

with ma the maximum age considered; similarly, let $\alpha = (\alpha_1, ..., \alpha_{ma})'$, $\beta = (\beta_1, ..., \beta_{ma})'$, and $\epsilon_t = (\epsilon_{1,t}, ..., \epsilon_{ma,t})'$. Then, using (2.17),

$$\begin{aligned} \ell_t &= \alpha + \beta\kappa_t + \epsilon_t \\ &= \beta c + (\alpha + \beta\kappa_{t-1} + \epsilon_{t-1}) + (\beta\delta_t + \epsilon_t - \epsilon_{t-1}) \\ &= \theta + \ell_{t-1} + \zeta_t \end{aligned} \tag{2.21}$$

[16] There are other sources of model risk as well. For instance, the Lee and Carter (1992) model class might be too small, not containing the actual distribution of $\widetilde{\kappa}_{T+t}$. This is also a source of model risk: we might require a more extensive or an other model class than the Lee and Carter (1992)-model class if we want to include the actual distribution of $\widetilde{\kappa}_{T+t}$. Possible other model classes, corresponding to much more (model) longevity risk, are discussed in the next section. However, the limited availability of mortality data makes it quite hard to determine whether the Lee and Carter (1992) model class is large enough or not. In this chapter, we focus on model risk within the Lee and Carter (1992)-model class.

with

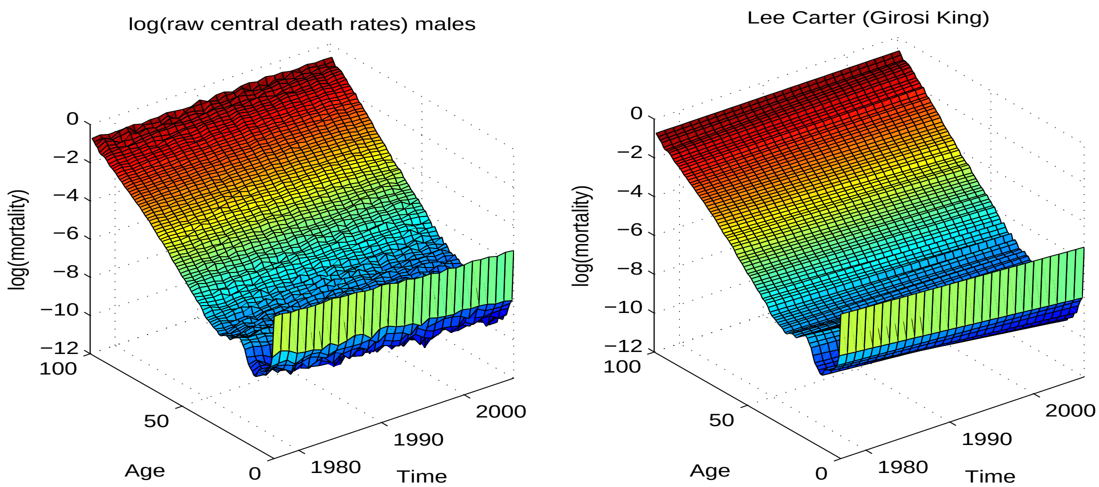
$$\theta = \beta c, \ \zeta_t = \beta \delta_t + \epsilon_t - \epsilon_{t-1}.$$

The Lee and Carter (1992)-model rewritten in this way can easily be estimated and used to make predictions and to quantify the longevity risk. For instance, with $\Delta \ell_t = \ell_t - \ell_{t-1}$, we can estimate θ simply by the time average of $\Delta \ell_t$, i.e., by

$$\widehat{\theta} = \frac{1}{T-1} \sum_{t=2}^{T} \Delta \ell_t = \frac{1}{T-1} \left(\ell_T - \ell_1 \right). \tag{2.22}$$

This estimator has well-known (T-asymptotic) characteristics, implying that making predictions as well as quantifying the longevity risk becomes a standard exercise in statistics or econometrics (both theoretically and practically). In Figure 2.3.1 the

Figure 2.3.1: **Log mortality and Lee-Carter fit** (Dutch males, using Girosi and King, 2006).

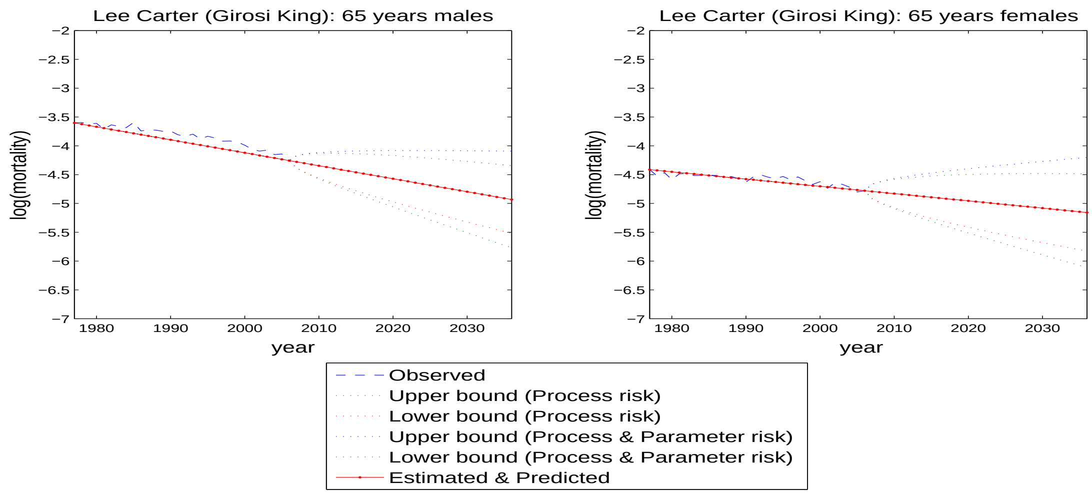


left panel shows the logarithm of the raw central death rates of Dutch males for ages 0 to 99 years and age class 100-110 years (indicated as age 100), over the sample period 1977 to 2006. This graph shows for each year the typical pattern of mortality as a function of age, starting rather high at age zero, revealing the level of infant mortality, then going down rather steeply to around age 10, and then increasing slowly, with a hump around age 20. This hump is typical for Dutch males, absent in the similar graph for Dutch females.[17] The right panel of Figure 2.3.1 shows the

[17]In case of other countries, this hump is typically observed for *both* males and females.

fitted values of the Girosi and King (2006)-variant of the Lee and Carter (1992) model, showing that this parsimonious model seems to be able to fit the mortality patterns observed in the data quite well.

Figure 2.3.2: **Predicting log mortality** (using Girosi and King (2006)-variant of Lee-Carter).



Next, we illustrate in Figure 2.3.2 the 30 year ahead prediction of the logarithm of the central death rates for 65 year old Dutch males and females, using the Girosi and King (2006)-variant of the Lee and Carter (1992) model. The prediction starts at the year 2007, the first year after the available sample period. In this figure we also include longevity risk, distinguishing between only process risk and the combination of process and parameter risk (in both cases 95% confidence intervals). The graphs show a clear estimated downward trend, both in-sample and predicted out-of-sample. In case of 65 year old males this trend corresponds to a decrease of the one-year death probability[18] of 0.0269 at the beginning of the sample (1977) down to 0.0141, predicted 30 years ahead, a decrease of almost 50%. In case of females, the one year death probability in 1977 equals 0.0120 and is predicted to go down to 0.0084, predicted 30 years ahead, a decrease of around 30%. However, these predictions are surrounded with substantial longevity risk (consisting of both process and parameter risk), including (with 95% confidence according to the model) no further decrease in mortality as well as a much more steeper decrease than during the sample period.

[18]Calculated using equation (2.13).

2.3.3 Recent Dynamic Mortality Models

The number of deaths is an integer-valued variable. Therefore, a Poisson process might be a more plausible way to model the number of deaths. Brouhns, Denuit, and Vermunt (2002a) model the integer-valued number of deaths $D_{x,t}$ as a Poisson distributed random variable,

$$D_{x,t} \sim \text{Poisson}\left(E_{x,t}\widetilde{m}_{x,t}\right), \tag{2.23}$$

with the systematic part of the central death rate $\widetilde{m}_{x,t}$ modeled as $\widetilde{m}_{x,t} = \exp\left(\alpha_x + \beta_x \kappa_t\right)$, comparable to the Lee and Carter (1992)-model. The model can be estimated following the same steps as in the original Lee and Carter (1992)-approach, but with the first step replaced by maximum likelihood using, for instance, the iterative method proposed in Goodman (1979). Brouhns, Denuit, and Van Keilegom (2005) discuss bootstrapping the Brouhns et al. (2002a)-model in order to quantify the longevity risk.

Cosette et al. (2007) propose as adjustment of the Lee and Carter (1992)-model to model the number of deaths as a Binomial process

$$D_{x,t} \sim \text{Bin}\left(E_{x,t}, q_{x,t}\right), \tag{2.24}$$

with $q_{x,t}$ modeled as $q_{x,t} = 1 - \exp\left(-\widetilde{m}_{x,t}\right)$, according to equation (2.13). The systematic part of the central death rate (or force of mortality) is again modeled in line with Lee and Carter (1992) as $\widetilde{m}_{x,t} = \exp\left(\alpha_x + \beta_x \kappa_t\right)$. This model can be estimated like the Brouhns et al. (2002a)-model, and the longevity risk can be quantified by means of bootstrapping.

The Lee and Carter (1992)-model implicitly assumes that there is no heterogeneity in the measurement error terms $\epsilon_{x,t}$, see (2.15). Li, Hardy, and Tan (2006) propose a way to incorporate heterogeneity into the Brouhns et al. (2000a)-variant of the Lee and Carter (1992)-model. Alternatively, Delwarde, Denuit, and Partrat (2007) suggest to use the Negative Binomial distribution to allow for more heterogeneity.

The Lee and Carter (1992)-model results in estimates for the parameters α_x and β_x for each given age x. Using α_x and β_x for each year of age might result in localized age induced anomalies. Lee and Carter (1992) proposed to have age groups ([0, 1), [1, 5), [5, 9) ..., [80, 85)), and in addition the age group [85, 109). Such age groups avoid localized age induced anomalies. However, this method leads to mortality rates that are equal for age groups of five years. Such an approximation might be quite crude, especially for valuating pension contracts. Renshaw and Haberman (2003b) propose to first estimate the parameters of the model using the one-year age groups and then to smooth using, for instance, a cubic spline. More recently, Delwarde, Denuit, and Eilers (2007) propose to smooth the β_x parameters as part of the first step, using a penalized log-likelihood approach in the Brouhns et al. (2002a)-variant of the Lee and Carter (1992)-model.

The Lee and Carter (1992)-approach also has some drawbacks. An important drawback follows from the reformulation by Girosi and King (2006). As follows from the estimator (2.22), see also Figure 2.3.2, the drift term of the random walk can be estimated by fitting a line for each age x through the first and final observation of the $\ln(m_{x,t})$ in the sample. Extrapolating these lines yields the age specific mid-points of the mortality projections (the "point estimates"). However, as long as the lines corresponding to different ages are not parallel, this implies that (very) long term mortality projections might become quite implausible, as is clearly illustrated in Girosi and King (2006), see also Girosi and King (2008). Their solution is to work with appropriate priors.

The problem of deviating long term forecasts might become even worse when the Lee and Carter (1992) methodology is applied to different groups g, each with its own specific process $\{\kappa_t^{(g)}\}$, representing the evolution of mortality over time. However, Wilson (2001) documents a global convergence in mortality levels. Li and Lee (2005) propose to adapt the Lee and Carter (1992)-approach by first identifying the central tendency, resulting in a common random walk with drift process $\{\kappa_t\}$, representing the joint evolution over time, and then to find the group specific stationary time processes $\{\kappa_t^{(g)}\}$, that represent the short term group g deviations from the common time trend.

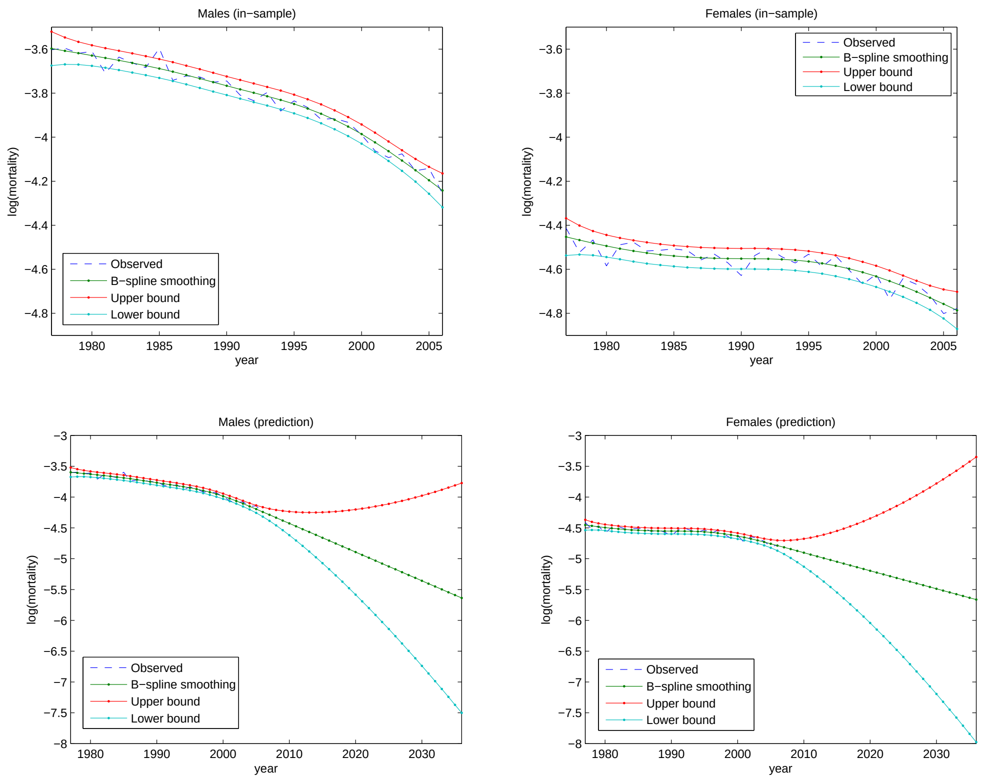
Finally, the Lee and Carter (1992)-model can only be used for groups for which sufficient data on mortality of different ages is available. Typically, this is an entire population of males and/or females of a country or a large region. However, the relevant population for an insurance company or a pension fund might deviate from the population for which data is available. For instance, Brouhns and Denuit (2001) and Denuit (2008) find that there is a significantly lower mortality rate for the group of insured individuals that were investigated compared with the whole male and female Belgian population. This might limit the applicability of the Lee and Carter (1992)-approach. Plat (2008) proposes a way to construct a portfolio-specific stochastic mortality model.

Next to the Lee and Carter (1992)-time series based stochastic mortality models, there are also other classes of time series based stochastic mortality models, for instance, imposing extra smoothness. Cairns, Blake, and Dowd (2006b) propose a model that builds in smoothness in mortality rates across adjacent ages in the same year, which is referred to as the "Cairns-Blake-Dowd (CBD) model." Cairns et al. (2009) propose generalizations of the CBD model to include a cohort effect, a quadratic term to the age effect, and a cohort which diminishes over time. Sweeting (2009) extends the original CBD model by allowing for a trend change in the parameters. Using England and Wales mortality data from 1841 till 2003 he observe 9 trend changes in the parameters. In the recent decades the trend changes are in the years 1945, 1973, 1988, and 1998. The impact of including a trend change on longevity risk is large, illustrated by a 90% confidence interval of the life expectancy for 60 year old males in 2056 of 18.7 years with trend change, parameter, and process uncertainty, only 7.6 years without trend change uncertainty but including parameter uncertainty and 3.6 years with only process uncertainty.

Currie, Durbin, and Eilers (2004) propose a model assuming smoothness across both ages and years. Cairns et al. (2008c-d, 2009) provide an extensive comparison of these various time series based stochastic mortality models.

To illustrate the effect of smoothing we present in Figure 2.3.3 an application of the Currie, Durbin, and Eilers (2004)-method in terms of the logarithm of the central death rate, using the same data as in case of Figures 2.3.1–2.3.2. The upper panels contain the in-sample results for 65 year old males (left) and females (right). These graphs show that the evolution of mortality over time has some curvature, which is captured quite well and in a smooth way by the Currie, Durbin, and Eilers (2004)-method (which employs so-called B-splines). Such a curvature will not be captured by the Lee and Carter (1992) model. In case of males there seems to be some acceleration in the decrease of mortality, while for females there is at first some slowing down and then again a slight acceleration in the decrease of mortality. The lower panels show the 30 years ahead predictions including 95% confidence intervals reflecting the longevity risk. The acceleration with regard to the males is translated into forecasts that are much lower than those derived from the Girosi and King (2006)-variant of the Lee and Carter (1992) model. In fact, 65 year old males and females are predicted to have more or less the same mortality characteristics 30 years from now. However, the longevity risk is quite substantial, leaving the possibility

Figure 2.3.3: **Smooth log mortality estimation and prediction** (using Currie, Durbin, and Eilers, 2004).



(with 95% confidence according to the model) of a wide variety of possible future mortality trends. The result of much wider prediction intervals, when changing the model from Lee and Carter (1992) to Currie, Durbin, and Eilers (2004), shows the importance of taking into account model risk.

2.4 Quantifying Longevity Risk

There are several studies that illustrate the importance of longevity risk for pension funds and insurance companies. The approaches differ both in terms of how longevity risk is quantified, and in terms of how the probability distribution of future mortality is modeled. For the former, we distinguish three approaches. First, an often used approach to quantify longevity risk in annuity portfolios is to determine its effect on the probability distribution of the present value of all future payments, for a given, deterministic, and constant term structure of interest rates (see, for example, Olivieri, 2001, Brouhns, Denuit, and Vermunt, 2002b, Dowd, Cairns, and Blake, 2006, and Cosette et al., 2007). Next, there is some literature that focuses on the effect of longevity risk on a pension fund's probability of underfunding (Olivieri and Pitacco, 2003, Hári et al., 2008b). Finally, longevity risk can be quantified by determining its effect on the probability of ruin for a portfolio of longevity-linked liabilities (Olivieri and Pitacco, 2003 and Chapter 5).

With regard to modeling the probability distribution of future mortality, several approaches discussed in the previous section are used, with the most popular among them the variants of the Lee and Carter (1992)-approach. For example, Brouhns et al. (2002a) use the variant with the Poisson distribution, and Cosette et al. (2007) use the variant with the Binomial distribution. Olivieri (2001) and Milevsky, Promislow, and Young (2006) instead present theoretical studies showing the implications of longevity risk in a setting where uncertainty in future mortality is modeled by means of three hypothetical scenarios. Other illustrations and references can be found in the review articles and monographs, mentioned at the beginning of the previous section.

In Sections 2.4.1, 2.4.2, and 2.4.3, we discuss the approaches in Olivieri (2001), Hári et al. (2008b), and Olivieri and Pitacco (2003), respectively. In each case, we consider a given and fixed date t, and quantify the effect of longevity risk on the liability payments in all future years.

Throughout this and the following section, we denote BEL_τ for the *best estimate value* of the liabilities at date $\tau \geq t$, which is defined as the market value of the liabilities in the best estimate scenario for future mortality development, i.e.,

$$BEL_\tau := \sum_{s \geq 1} \mathbb{E}_\tau \left[\widetilde{L}_{\tau+s} \right] \cdot P_\tau^{(s)}, \tag{2.25}$$

where $\widetilde{L}_{\tau+s}$ denotes the liability payment at time $\tau + s$, $P_\tau^{(s)}$ denotes the date-τ market value of a zero-coupon bond maturing at time $\tau + s$, and $\mathbb{E}_\tau[\cdot]$ denotes the expectation, conditional on death rates up to time τ.

2.4.1 Discounted Present Value of Liabilities

In this section we discuss the analysis in Olivieri (2001), who focusses on the relative importance of individual mortality risk and longevity risk. She quantifies longevity risk in annuity portfolios by determining its effect on the probability distribution of the present value of future payments. The one-year death probabilities are assumed to follow the mortality law of Heligman-Pollard. Olivieri (2001) incorporates longevity risk by considering three possible future scenarios (a worst case, a medium case, and a best case). In this section, we replicate her results, but instead of assuming three possible scenarios in terms of the Heligman-Pollard mortality law, we model the uncertainty in the probability distribution of future mortality following Appendix 5.B.1. This means that we include process and parameter risk in the future death probabilities on the basis of the Lee and Carter (1992)-approach. In addition, we shall allow for uncertainty in the model-variant choice: We include next to the traditional Lee and Carter (1992)-model also the variants proposed by Brouhns et al. (2002a) and Cosette et al. (2007). In this way we also allow for model risk.[19]

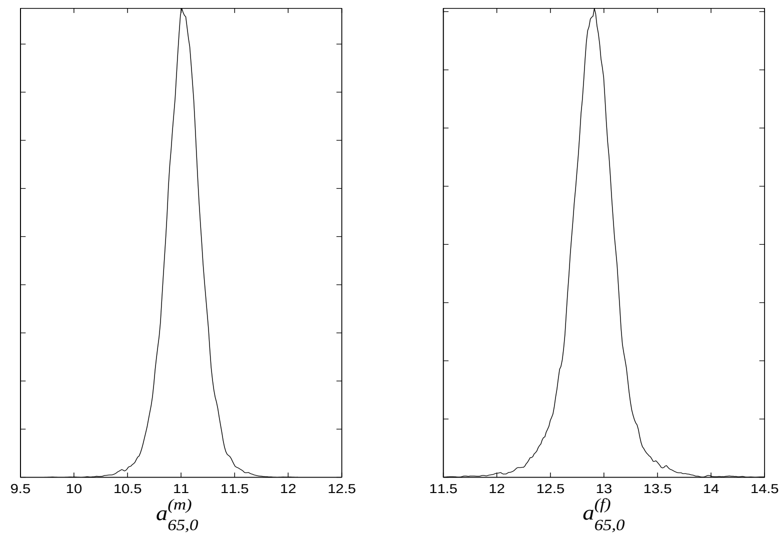
For a given and fixed year t, we consider the present value of all future payments in a portfolio of pension annuities. There are N annuitants, all of age $x = 65$ at time t. In our case t corresponds to the year 2006. The annuity pays off one Euro every year that the annuitant survives. The time-t present value of the annuity payments to annuitant i, denoted by Y_i, is defined in (2.8), where we shall assume a constant annual interest rate, equal to $r = 0.04$. Conditional upon the one year death probabilities after time t, we can calculate the expected value of the present value Y_i, for a given i, resulting in $a_{65,t}^{(m)}$ for the Dutch male and in $a_{65,t}^{(f)}$ for the Dutch female, where $a_{x,t}^{(g)}$ is defined in equation (2.9). Without longevity risk, this expectation would be the fair value of the annuity. In Figure 2.4.1 we present the distributions of $a_{65,t}^{(m)}$ and $a_{65,t}^{(f)}$, when the future death probabilities are random as described above (including process, parameter, and model risk). The distribution for females (around just below 13 Euro) is shifted to the right compared to the distribution of males (around 11 Euro). This reflects the fact that females, on average, become older than males. Moreover, the figure clearly reveals substantial longevity risk in the annuities, implying that a fair valuation might require a substantial risk premium.

Next, we reproduce Table 5 of Olivieri (2001). The present value of the portfolio of annuities is given by $\mathcal{Y} = \sum_{i=1}^{N} Y_i$. We shall assume that conditional upon $\mathcal{F}_\infty = \left\{ q_{x,t+\tau}^{(g)} \mid \tau \geqslant 0 \right\}$ the Y_i are distributed independently. Table 2.4.1 presents our results.

The first row reports the best estimates of the annuity portfolio, for both males and females for different sizes N. For $N = 1$ it yields the best estimate for the

[19] For a detailed description we refer to the Appendix of Chapter 5.

Figure 2.4.1: **Distribution Annuity Portfolio**



This figure presents the distribution of the annuity portfolio at time $t = 0$ (corresponding to the year 2006) due to longevity risk only (i.e., after pooling). For all annuitants the age is $x = 65$. The left panel applies to a portfolio of males, the right panel to a portfolio of females.

Table 2.4.1: **Descriptive Statistics Annuity Portfolio**

	Males			Females		
	$N = 1$	$N = 100$	$N = 1000$	$N = 1$	$N = 100$	$N = 1000$
$\mathbb{E}(\mathcal{Y})$	11.024	1102.365	11023.654	12.897	1289.737	12897.374
$\mathbb{E}(\mathrm{Var}(\mathcal{Y}\vert\mathcal{F}_\infty))$	19.604	1960.397	19603.968	17.963	1796.332	17963.316
$\mathrm{Var}(\mathbb{E}(\mathcal{Y}\vert\mathcal{F}_\infty))$	0.031	312.609	31260.878	0.057	567.703	56770.266
$\mathrm{Var}(\mathcal{Y})$	19.635	2273.006	50864.845	18.020	2364.034	74733.582
γ	0.40197	0.04325	0.02046	0.32914	0.0377	0.0212

annuity. The next three rows present the variances of these portfolios, together with a decomposition as in equation (2.11):

$$\mathrm{Var}\left(\mathcal{Y}\right) = \mathbb{E}\left(\mathrm{Var}\left(\mathcal{Y} \mid \mathcal{F}_\infty\right)\right) + \mathrm{Var}\left(\mathbb{E}\left(\mathcal{Y} \mid \mathcal{F}_\infty\right)\right). \tag{2.26}$$

The first term on the right hand side of this equation corresponds to the portfolio risk if there would be no longevity risk. This risk increases linearly in N, since $\mathbb{E}\left(\mathrm{Var}\left(\mathcal{Y} \mid \mathcal{F}_\infty\right)\right) = N\mathbb{E}\left(\mathrm{Var}\left(Y_i \mid \mathcal{F}_\infty\right)\right)$. The second term on the right hand side of (2.26) is due to the presence of longevity risk. This term increases by N^2 with increasing portfolio size N, since $\mathrm{Var}\left(\mathbb{E}\left(\mathcal{Y} \mid \mathcal{F}_\infty\right)\right) = N^2\mathrm{Var}\left(\mathbb{E}\left(Y_i \mid \mathcal{F}_\infty\right)\right)$. Thus, for larger portfolios this term will dominate the total portfolio risk. This can also be seen from the results presented in the table.

The final row of the table presents the coefficient of variation of $\mathcal{Y}$, defined by $\gamma = \sqrt{\mathrm{Var}\left(\mathcal{Y}\right)}/\mathbb{E}\left(\mathcal{Y}\right)$. This coefficient allows a better investigation of the size of the portfolio on its riskiness. Without longevity risk, this coefficient vanishes with increasing portfolio size N, due to the pooling effect. However, with longevity risk, we get

$$\gamma = \left(\frac{1}{N}\frac{\mathbb{E}\left(\mathrm{Var}\left(Y_i \mid \mathcal{F}_\infty\right)\right)}{\mathbb{E}\left(Y_i\right)} + \frac{\mathrm{Var}\left(\mathbb{E}\left(Y_i \mid \mathcal{F}_\infty\right)\right)}{\mathbb{E}^2\left(Y_i\right)}\right)^{1/2}, \tag{2.27}$$

showing that for large portfolio sizes N indeed the longevity risk dominates the total risk, and also does not disappear. In our example, the limiting value of the coefficient of variation equals $\gamma = 0.0160$ for males and $\gamma = 0.0185$ for females.

Olivieri (2001) also calculates the boundary portfolio size $\overline{N}$ such that for portfolio sizes larger than this bound longevity risk dominates the total risk. She calculates this bound as

$$\overline{N} = \frac{\mathbb{E}\left(\mathrm{Var}\left(\mathcal{Y} \mid \mathcal{F}_\infty\right)\right)}{\mathrm{Var}\left(\mathbb{E}\left(\mathcal{Y} \mid \mathcal{F}_\infty\right)\right)}. \tag{2.28}$$

In our case (i.e., with survival probabilities forecasted with the Lee and Carter-methodology), the boundary value is $\overline{N} = 628$ for males and $\overline{N} = 317$ for females. Although substantially larger than the $\overline{N} = 12$ reported by Olivieri (2001), these numbers are quite low, indicating that also for smaller portfolio sizes longevity risk is an important risk that should be taken into account in a risk management framework.

2.4.2 Funding Ratio Volatility

A drawback of the approach described in Section 2.4.1 is that it is a "liability only" approach: it ignores the potential impact of financial risk on the importance of longevity risk. Therefore, in this section we discuss an alternative approach in which the importance of longevity risk is quantified by determining its effect on the probability distribution of the funding ratio at a future date (see, for example, Olivieri and Pitacco, 2003, and Hári et al., 2008b). The funding ratio is defined as the value of the assets divided by the value of the liabilities. Determining the value of longevity-linked liabilities, however, is still a contentious issue. There is extensive literature on the pricing of longevity-linked liabilities (see, e.g., Bauer, Boerger, and Russ, 2010), but due to the high degree of illiquidity and market incompleteness, it remains difficult to calibrate these pricing models. Therefore, the regulator requires that the liabilities should be valued at their *fair value*.

Hári et al. (2008b) use a simulation analysis to determine the distributional characteristics of the funding ratio at the beginning of year $t + T$, for maturities $T = 1$ and $T = 5$, respectively, given that the funding ratio in year t equals 1. They consider a pension fund with N annuitants at the beginning of year $t = 2004$, and quantify the uncertainty in future funding ratios for various investment strategies. In order to illustrate the effect of portfolio size, they consider portfolios of different sizes. In each case, the age and gender composition of the pension fund is the portrayal of the Dutch population at the beginning of 2004. They use a *run-off approach* (i.e., they consider a setting where there are no new entrants into the fund, and no rights are built up or premiums are paid after time t), and let the fair value of the liabilities be given by the best estimate value, as defined in (2.25).[20] They quantify the funding ratio volatility using five investment strategies: i) liabilities are "perfectly" hedged: expected liabilities are hedged with cash-flow matching initially; ii) liabilities are duration hedged, based on the McCauley duration; iii) assets are invested exclusively in 5-year bonds; iv) 50% of the assets is invested into 5-year and 50% in 10-year bonds; v) 37.5% is invested into 5-year, 37.5% in 10-year bonds, and the rest is invested into stocks; vi) 25% is invested in 5-year, 25% in 10-year bonds, while the rest is invested in stocks. Their main findings are as follows:

- As the fund size increases, individual mortality risk in relative terms decreases to zero, due to the pooling effect. In contrast, longevity risk does not become

[20] This is in line with Dutch solvency regulations at the time the research was performed.

negligible; it is almost independent of portfolio size.

- If financial market risk is perfectly hedged (so that uncertainty in future lifetime is the only source of risk), then pension funds are exposed to a substantial amount of uncertainty. For instance, for a large fund (10,000 participants), the standard deviation of the funding ratio in a 5-year horizon is then 5.3% of the expected value.

- If financial market risk is also considered, the contribution of longevity risk to the overall risk becomes less important. However, whenever the investment strategy is not too risky, longevity risk is likely to remain significant.

2.4.3 The Ruin Probability

The approaches discussed in the previous two sections each have their drawbacks. First, as argued before, quantifying the uncertainty in the discounted present value of liability payments for a given and deterministic interest rate, as in Section 2.4.1, is a liability only approach that ignores the effect of a pension fund's investment strategy on the impact of longevity risk. In contrast, a funding ratio approach takes into account both assets and liabilities. However, quantifying the uncertainty in the funding ratio, as discussed in Section 2.4.2, requires making assumptions regarding the fair value of longevity-linked liabilities. As discussed in Section 2.2.3, it is unlikely that the price of longevity-linked liabilities equals the best estimate value. The best estimate is likely to be an underestimate of the price at which the pension fund could sell its liabilities. One might argue that this problem could be mitigated by adding a *market value margin* to the best estimate value of the liabilities, as suggested by Solvency II. However, when the market value margin does not accurately reflect the risk premium that a third party would require in order to be willing to take over the liabilities, it remains unclear to what extent the funding ratio approach accurately quantifies longevity risk.

In this section we discuss an alternative approach to quantify longevity risk, namely by determining its effect on the probability of ruin (see, for example, Olivieri and Pitacco 2003, and Chapter 4). We also discuss how this approach relates to, and differs from, the approaches described in the previous two sections.

The ruin probability approach, in a run-off approach in which there are no new entrants into the fund, and no rights are built up or premiums are paid after time t, is extensively described in Chapter 4. In this approach longevity risk is be quantified by determining the minimum level of the asset value at time t that is required in order to limit the probability of a negative terminal asset value to ϵ. The minimum level of the asset value at time t is given by the the 1-ϵ percentile of L_t, with

$$L_t = \sum_{s=1}^{\mathcal{T}} \frac{\widetilde{L}_{t+s}}{(1 + r_t^{(s)})^s}, \tag{2.29}$$

where $r_t^{(s)}$ denotes the annualized portfolio return over the period $[t, t+s]$ and $t+\mathcal{T}$ denotes the last period in which liabilities needs to be paid. This allows for the following comparison:

- when the asset portfolio consists of one-year bonds and there is no interest rate uncertainty, then $r_t^{(\tau)} = r$, and $L_t = \sum_{s=1}^{\mathcal{T}} \frac{\widetilde{L}_{t+s}}{(1+r)^s}$. Thus, under the assumption that the pension fund will earn a minimal return of r on its investments, the $(1-\epsilon)$-quantile of the discounted present value of liability payments for a constant and deterministic interest rate r, as described in Section 2.4.1, can be interpreted as the level of assets that is sufficient to guarantee that the probability of ruin is below ϵ.

- Whereas the funding ratio approach described in the previous section amounts to comparing, *at a given time* $t+T$, the value of the assets to the *fair value* of the liabilities, the ruin probability approach is equivalent to requiring that the asset value at time t combined with any future returns on these assets is sufficient to cover the *actual* liabilities in each future year.

2.5 Illustrating Longevity Risk Management

2.5.1 Longevity Risk Management

As illustrated in the previous section, longevity risk can be substantial for life insurers or pension funds. Likely, this risk factor is not the most important one faced by a life insurer or pension fund, but, given its significance, it cannot be ignored. The typical approach to deal with the effects of changes in mortality rates on pension and insurance liabilities has long been to re-estimate these rates on a regular basis, and to recalculate the value of the liabilities accordingly. Although this accounts to some extent for changes in survival, it is a retrospective approach. It does not take into account future changes in mortality, and thus ignores longevity risk. Instead, a modern risk management approach requires to manage longevity risk, just like other risk factors, in an effective way, see, for instance, Pitacco (2007) or Cairns, Blake, and Dowd (2008a). Following Cairns et al. (2008a), there is a range of possibilities to deal with longevity risk.

- Life insurers and pension funds might retain longevity risk as part of their business risk. This would require an appropriate asset liability management (ALM) to guarantee that the assets suffice to meet the liabilities. As an illustration, we discuss in Chapter 3 the determination of solvency buffers needed to reduce the probability of underfunding of a pension fund or insurance company to an acceptable level.

- Life insurers and pension funds might enter into a variety of forms of reinsurance, or they might arrange a (full or partial) buyout of their liabilities by a specialist insurer. Cairns, Blake, and Dowd (2008b) discuss this traditional possibility in some detail. See also Biffis and Blake (2010).

- Life insurers and pension funds might try to diversify longevity risk, in particular, using different products. Sometimes, natural hedges exist, see, for example, Milevsky and Promislow (2001) or Cox and Lin (2007). We illustrate the diversification possibilities through product mix in Chapter 5. Related to this, in order to share the longevity losses or benefits, life insurers and pension funds might develop new products with adjustable starting dates or payments depending on realized life expectancy.

- Life insurers and pension plans might try to securitize part of their business, or they might try to manage their longevity risk using mortality-linked derivatives. In Section 2.5.2 we discuss these possibilities further.

2.5.2 Securitization and Mortality-Linked Derivatives

While redesign of pension and insurance deals as illustrated in the previous section can mitigate the effects of mortality risk to some extent, this risk will never be eliminated completely. Therefore, uncertainty regarding future survival rates will continue to impose risk on any pension fund or life insurance company.

The introduction of financial instruments for which the payoff is linked, to some extent, to the development of mortality rates could help insurers and pension funds manage their risk. Loeys, Panigirtzoglou, and Ribeiro (2007) investigate whether a "life market," where such products might be traded, could be successful. They explain that for a new capital market to be established and to succeed, "it (1) must provide *effective exposure*, or hedging to a state of the world that is (2) *economically important* and that (3) *cannot be hedged through existing market instruments*, and (4) it must use a *homogeneous and transparent contract* to permit exchange between agents." They argue that "longevity meets the basic conditions for a successful market innovation." Blake, Cairns, and Dowd (2008b) investigate the conditions suggested by Loeys et al. (2007) in more detail. These authors maintain that there is insufficient reinsurance capacity to deal with global longevity risk, while capital markets are more efficient than the insurance industry in reducing informational asymmetries and in facilitating price discovery. This makes them confident that a fully developed capital market will emerge soon.

One of the first attempts to set up such a market is a standard coupon-plus-principal bond for which the coupon is determined by the term structure of interest rates, but the principal of the bond depends on the extent to which the actual observed survival in a predefined population, measured by a "survival index," exceeds a given threshold level (see Blake and Burrows, 2001, and Blake et al., 2006). By investing in such *longevity bonds*, the risk of higher than expected survival can be partially transferred to the issuer of the bond. The European Investment Bank (EIB) together with BNP Paribas issued a longevity bond in 2004, but there was too little demand to reach a level adequate enough to sustain in a market.[21] The high degree of market incompleteness implies that pricing this product is nontrivial. This might explain why longevity bonds have not yet been successfully introduced in the market. See Blake et al. (2008b) for an extensive investigation of this failure.

Indeed, the potentially severe consequences of underpricing the risk may hamper the introduction of longevity linked securities. For example, inaccurate pricing of the risk in guaranteed annuity options induced by uncertain changes in interest rates, led to the downfall of the large British insurance company Equitable in 2000 (see Pelsser, 2003). The current literature devotes considerable attention to this pricing problem. Specifically, traditional finance approaches (risk-neutral pricing theories, see, for example, Cairns et al., 2006a) as well as actuarial pricing approaches (Wang's

[21]On the other hand, the issue of short-dated mortality bonds, which are similar to catastrophic bonds, has been successful. However, such bonds hedge against catastrophic mortality risk, not longevity risk.

premium principle, see, for example, Lin and Cox, 2005) have received considerable attention. However, market incompleteness implies that calibrating these pricing models remains difficult.

Another explanation for the failure of the longevity bond is the effect of basis risk, i.e., the difference in the survival probabilities between the insured population of the annuity provider and the national population which is used for the payouts of the longevity bond. This basis risk effects the hedge effectiveness of standard longevity risk transfer products. Coughlan et al. (2010) show that the effect of basis risk is small, when hedging the value of a 10-year deferred annuity for a 55 years old male insured with insured mortality using a 10-year deferred annuity swap that pays out on the basis of the survival index for the national population for 55-years old males. In Chapter 5 we show that basis risk can be substantial when hedging longevity risk in a portfolio of life insurance products for insured aged 65, using a longevity swap that yearly pays out on the basis of the survival index for the national population for 65-years old males and/or females.

Blake, Boardman, and Cairns (2010) propose that governments should issue a deferred tail longevity bond with payments, for a currently 65-year old, starting from age 90. Capital markets should deal with longevity risk covering the age ranges 65-90. By letting the capital markets only covering longevity risk between ages 65 and 90 it can overcome some of the problems in the longevity risk transfer market, such as the long duration of the bond and the "toxic tail", i.e., the large uncertainty in survival probabilities at advanced ages in the long run. There is growing support for the government to issue longevity bonds, for example the IMF (Groome et al., 2006), OECD (Antolin and Blommestein, 2007 and Antolin, 2008), and the World Economic Forum (Hayashi et al., 2010) all recognize that the government can improve the longevity risk market by issuing longevity indexed bonds.

An alternative and more successful attempt to deal with longevity risk is securitization.[22] In this case, a pool of assets or liabilities is sold to a so-called Special Purpose Vehicle. These assets or liabilities are then repackaged as new securities, and as such traded in the capital market. Blake et al. (2008b) discuss the different types of securitization with longevity-linked assets or liabilities, known as insurance-linked securities (see Krutov, 2006). Cowley and Cummins (2005) discuss the earlier types of securitization.

In order to encourage the development of a "life market" JPMorgan introduced

[22] We follow Blake et al. (2008b).

in March 2007 so-called "longevity indices." The idea of introducing such indices is that this objective information provided by the indices might stimulate the introduction and subsequent trade of mortality-linked securities. In December 2008 Deutsche Börse introduced longevity indices called "Xpect Indices", which are two different index types, providing information on life expectancy and mortality risks in Germany, the Netherlands and later also England and Wales, broken down according to region, year of birth and gender. The "Xpect Age Indices" represent the average remaining life expectancy of a defined age and gender group and the "Xpect Cohort Indices" represents the number of survivors starting from a defined date with an initial value of 100,000. In February 2010 the "Life and Longevity Market Association" (LLMA), a cross-industry, non-profit association was launched. The LLMA promotes the development of a liquid traded market in longevity and mortality related risk by development of consistent standards, methodologies, benchmarks and best practice. In addition, in August 2010 the LLMA published a draft "Longevity Index Framework".[23]

The mixed success thus far of initiating a life market has generated several proposals to set up such a market using mortality-linked *derivatives*. Mortality and survivor swaps are an example of such derivatives. In case of such a swap one party pays fixed payments to the other party in exchange for payments that depend on the number of people in a given cohort that die in a given period (mortality swap) or that survive during that period (survivor swap). See Dowd, Blake, Cairns, and Dawson (2006) or Dawson, Blake, Cairns, and Dowd (2007). Another example can be found in mortality and longevity forwards. In this case the contract involves the exchange of a payment depending on the realized mortality or survival rate at the maturity of the contract in return for a payment depending on a fixed mortality or survival rate agreed upon at the initiation of the contract. See Blake et al. (2008b) for a further discussion and illustration.

Once a market for mortality–linked instruments arises, individual pension funds and insurers can use these instruments to hedge or reduce their risk. The asset portfolio should be designed in order to yield an optimal risk-return trade-off. See, for instance, Haberman and Vigna (2002) or Gerrard, Haberman, and Vigna (2004). A concern here is that optimization of risk-return trade-offs is known to be highly sensitive to parameter estimation if parameter uncertainty is ignored (for example,

[23]The 10 members in September 2010 are: AXA, Deutsche Bank, J.P. Morgan, Legal & General, Morgan Stanley, Pension Corporation, Prudential, RBS, Swiss Re, UBS.

Best and Grauer, 1991, or Chopra and Ziemba, 1993). This is clearly undesirable if the optimal portfolios with respect to the estimated parameter values lead to significantly suboptimal values of the objective function compared to the optimal portfolios for the true parameter values. In particular, this is a major concern in a setting in which mortality-linked assets and liabilities are involved, since the probability distribution of their value depends on very long-term forecasts. This requires the development of robust portfolio optimization techniques for Asset Liability Management of an individual pension fund or insurer in the presence of mortality risk.

2.6 Conclusions

This chapter provides an overview of the literature on longevity risk, i.e., the uncertainty in future changes in mortality rates. The chapter focus on the current models to forecast the distribution of future mortality probabilities, the different methods to quantify longevity risk and possibilities to deal with longevity risk as an insurer or pension fund.

We briefly describe three classes of models to forecast the distribution of future mortality probabilities, namely the Lee-Carter (1992)-model with extensions and modifications, the Cairns-Blake-Dowd model with extensions and modifications, and the P-Splines model. We illustrate the importance of taking various sources of risk, namely process, parameter, trend-change, and model risk, into account when forecasting the distribution of future survival probabilities.

We illustrate the importance of longevity risk for pension funds and life insurers, using three different methods. We discuss the method using discounted present value of liabilities. Using this method we show that the systematic longevity risk is more important than the non-systematic longevity risk even for portfolios with relatively few insureds. We further discuss the quantification of longevity risk in portfolios of life insurance products through the funding ratio volatility and through the ruin probability and we discuss the advantages and disadvantages of each method.

Finally, we illustrate some aspects of longevity risk management, for example, the determination of solvency buffers, and the effect of the product mix as a natural approach to diversifying longevity risk. Another approach is to securitize longevity risk or to use mortality linked derivatives. Although there have been attempts to set up a "life market," a trading place for mortality-based products, these initiatives have thus far been only partially successful. However, Loeys et al. (2007) argues that "longevity meets the basic conditions for a successful market innovation". As discussed by, among other, Cairns et al. (2008b) the government might assist by encouraging and facilitating the development of this market. In particular, the government could issue longevity bonds in order to establish a default-free term structure for longevity risk, similar to its activity in the fixed-income market.

Chapter 3

Calculating Capital Requirements

This chapter is based on Stevens, De Waegenaere and Melenberg (2010c).

3.1 Introduction

Over the last decades, significant improvements in the duration of life have been observed in most countries. For example, over the past three decades, the remaining life expectancy of a male Dutch retiree aged 65 has increased by on average one year per decade. However, there is considerable uncertainty regarding the future development of life expectancy, referred to as (systematic) longevity risk. Whereas the focus of regulators has long been on the risk in financial investments, there is now increasing awareness that accurate quantification and management of the risk in pension and insurance liabilities (including longevity risk) is equally important. For example, the goal of the Swiss Solvency Test and the Solvency II project (Group Consultatif Actuariel Europeen, 2006) is to redesign financial regulation of insurance companies in Switzerland and the EU, respectively, putting increased emphasis on the valuation and management of pension and insurance liabilities. Specifically, in Solvency II the regulator will require that an insurer holds a reserve in order to limit its probability of underfunding within one year to 0.5%, where an insurer is underfunded in case the value of the assets is less than the value of the liabilities.

In this chapter we develop a methodology in line with the Solvency II proposal to determine reserve requirements for systematic longevity risk in life insurance products. A complicating factor of reserve requirements for systematic longevity risk is that there is no liquid market for longevity-linked assets or liabilities, and so no market price is observed. Hence, the value of the liabilities needs to be calculated using a mark-to-model approach. In Solvency II the (fair) value of the liabilities is defined as the sum of the *best estimate value of the liability* (BEL), and a *market value margin* (MVM). The latter component can be interpreted as a risk premium, and should be determined following a *Cost of Capital* (CoC) approach. The idea is that the risk premium for a risky liability is determined by the amount of capital (the *solvency capital requirement*, SCR) the holder of the risk should hold in order to be able to pay the liabilities with a high degree of certainty. Our goal in this chapter is twofold. First, it has been argued extensively (see, Ulm, 2009, and Olivieri and Pitacco, 2008a) that determining capital requirements in line with the Solvency II proposal as described above is technically complex. The main complication is that the value of the liabilities in any given period depends on the required solvency capital, which, in turn, depends on the value of the liabilities as well as the probability distribution of the value of the liabilities in the next period. This implies that backward induction is needed to determine the current value of the liabilities. In addition, the value of the liabilities in a future period $t + s$ depends on the probability distribution of realized death probabilities as defined in (2.4), conditional on information available at time t. There is a wide variety of mortality forecast models that can be used to simulate the probability distribution of realized survival rates, see, for example, Lee and Carter (1992), Renshaw and Haberman (2006), Cairns, Blake, and Dowd (2006), Currie, Durban, and Eilers (2004). However, when applying such models the number of simulations needed in a backward induction algorithm to achieve a sufficiently accurate solution increases exponentially in the length of the run-off period, which can typically be very long for life insurance products. This makes a simulation approach computationally intractable. To deal with this problem, we develop a closed form approximation for the probability distribution of future mortality rates, conditional on information available at that time, starting from the widely used Lee-Carter (1992)-model. This closed form approximation enables us to determine the solvency capital requirements, with a possibly non-zero risk margin when calculating these requirements, within reasonable time with only a small approximation error.

Second, the regulator allows the use of simplified approaches that avoid recursive evaluations, by setting the risk margin equal to zero, when determining the solvency capital requirements. The value of the required solvency capital at future dates is then typically approximated by some current estimate. For instance, in the

Solvency II standard model the solvency requirement for longevity risk is computed using a longevity shock consisting of a 25% (QIS 4) or 20% (QIS 5) permanent reduction in all one year best estimate death probabilities. By comparing our results to those that follow from the simplified approaches of the regulator, we quantify the impact of simplifying assumptions on the capital requirements. In particular, we find that the 20% longevity shock leads to substantially higher capital reserves than our Lee-Carter based internal model. Other simplifying assumptions to determine the capital requirements generally tend to underestimate the capital requirements for the investigated portfolios. For example, the *market value margin* for a single life annuity in the UK for females is 2.04% of the best estimate of the liabilities in the internal model, but this value drops to only 0.77% of the best estimate of the liabilities when using the proposed simplifying assumption of a constant fraction of solvency capital requirements relative to the best estimate of the market value of the liabilities to determine the market value margin.

A closely related chapter is Börger (2010). This chapter also determines the *market value margin* and the *solvency capital requirement* for various life insurance products using the Cost of Capital method. However, the focus of our chapter is different from Börger (2010). Our aim is to calculate the solvency capital requirements, including a possibly nonzero *market value margin* as risk premiumwhen calculating the requirements. We are able to do so by using an (reasonably close) approximation, which we are able to obtain starting from the Lee and Carter (1992) modeling approach.

The remainder of the chapter is organized as follows. In Section 3.2 we discuss the Solvency II proposal for minimum capital requirements. In Section 3.3 we discuss the proposed simplifications to calculate the capital requirements in the Solvency II proposal. In Section 3.4 we shortly discuss the Lee-Carter (1992)-model and we present the closed form approximation of the distribution of the liabilities (with the details presented in the appendix). In Section 3.5 we calculate the capital requirements for different portfolios of life insurance products, using both the internal model as well as the simplifications. Section 3.6 concludes.

3.2 Direct approach of Solvency II

In this section we describe the method to determine the value and reserve requirement for a liability in the Cost of Capital approach. The Cost of Capital approach is based on the net asset value, which is described in Subsection 3.2.1. Using the net asset value we determine the value of the liabilities and the capital requirements for the single-period case in Subsection 3.2.2. In Subsection 3.2.3 we generalize the results to a multi-period setting.

3.2.1 Net asset value approach

The underlying idea of the Solvency II directive proposal is that insurers should hold an amount of capital that enables them to absorb unexpected losses and meet the obligations towards policy-holders at a high level of equitableness. The calculation of this requirement is to be made on the basis of the Value at Risk (VaR) calculation at the $1-\alpha$ percentile (in Solvency II α is set at 0.5%) for the time period of one year. Specifically, the regulator requires that the capital held by an insurer is such that the probability that the net asset value (NAV) falls below 0 within a year is lower than α, where the NAV is defined as the difference between the value of the assets (A_{t+1}) and the value of the liabilities (L_{t+1}), i.e., the NAV in year $t+1$ is given by:

$$NAV_{t+1} = A_{t+1} - L_{t+1}.$$

The NAV in year $t+1$ is a random variable at time t, because both the market value of the assets at time $t+1$ and the market value of the liabilities at time $t+1$ are random variables at time t. The Solvency II requirements imply that an insurer should hold at least initial assets with value $A_t^\star$, defined as:

$$A_t^\star \equiv \min \left\{ A_t | \mathbb{P}_t \left[NAV_{t+1} < 0 \right] \leq \alpha \right\}, \tag{3.1}$$

where $\mathbb{P}_t(\cdot)$ denotes the time-t probability distribution which represents all possible sources of risk, conditional upon the information available at time t. The required capital in excess of the value of the liabilities is referred to as the *Solvency Capital Requirement*[1] (SCR), i.e.,

$$SCR_t \equiv A_t^\star - L_t. \tag{3.2}$$

The minimum required capital $A_t^\star$ and the buffer SCR_t depend on the evolution of the assets over time. Let r_t be the (stochastic) asset portfolio return between time

[1] In Solvency II the SCR for life underwriting risk is decomposed into seven different risk factors, including longevity risk. The seven different risk factors are: revision risk, mortality risk, longevity risk, disability risk, lapse risk, expense risk, and catastrophe risk. In this chapter we focus on the effect of longevity risk.

t and $t+1$ and $\widetilde{L}_t$ be the time-t specific (aggregate) payment of the life insurance products at the end of year t, then the value of the assets next year is given by:

$$A_{t+1} = A_t \cdot (1 + r_t) - \widetilde{L}_t,$$

which implies that:

$$\mathbb{P}_t \left(NAV_{t+1} < 0 \right) = \mathbb{P}_t \left((1 + r_t) \cdot A_t - \widetilde{L}_t < L_{t+1} \right). \tag{3.3}$$

It follows from (3.2) and (3.3) that the required solvency capital is given by:

$$SCR_t = \mathbb{Q}_{1-\alpha,t} \left[\frac{\widetilde{L}_t + L_{t+1}}{1 + r_t} \right] - L_t, \tag{3.4}$$

where $\mathbb{Q}_{1-\alpha,t}[X]$ is the $1-\alpha$ percentile of the random variable X, whose distribution is induced by $\mathbb{P}_t$.

In order to be able to determine the required solvency capital in year t, we need the value of the liabilities in year t, as well as the probability distribution of the value of the liabilities in year $t+1$. In absence of a liquid market, there is no obvious unique way to determine the market values of the liabilities. Solvency II proposes to determine liability values on the basis of a mark-to-model approach, in which the value consists of the *Best Estimate of the Liabilities* (BEL) plus a *Market Value Margin* (MVM), where the latter is seen as a risk premium, i.e., for every t we have

$$L_t \equiv BEL_t + MVM_t. \tag{3.5}$$

The best estimate value is defined as the expected present value of all future payments, i.e.,

$$BEL_t \equiv \sum_{s \geq 0} \mathbb{E}_t \left[\widetilde{L}_{t+s} \right] \cdot P_t^{(s)}, \tag{3.6}$$

where $\mathbb{E}_t[\cdot]$ is the expectation with respect to $\mathbb{P}_t$, and $P_t^{(s)}$ denotes the time t price of a zero coupon bond that matures at time $t+s$.

The market value margin is intended to reflect the cost associated with holding the required solvency capital in the current and any future period.[2] The idea is that because the return on assets that needs to be kept as a reserve is generally lower

[2]See QIS 4, TS.II.A.29 and QIS 5, V.10.ii.

than the return on "free assets," the holder of a risky liability requires a price of taking the risk as a compensation for not being able to invest the reserve as a free asset. For the calculation of the value of the liabilities with a non-tradeable risk, such as longevity risk, the return on the assets in Solvency II is set equal to the risk-free return. Hence, the stochastic asset portfolio return r_t in equation (3.3) is replaced by the risk-free return of appropriate maturity. We shall proceed under the assumption of a flat term structure, where the one-period risk free rate is denote by r^{rf}. The intention of the regulator is then that the market value margin equals the cost, i.e., $CoC \times 100\%$, charged to the present value of the current and all future values of SCR_t, i.e.,

$$MVM_t = CoC \cdot \sum_{s \geq 0} \frac{SCR_{t+s}}{(1 + r^{rf})^s}. \tag{3.7}$$

However, this way of defining the market value margin is not feasible, since it leads to an unknown value at time t: the future values SCR_{t+s} of the required solvency capital are unknown at time t.

3.2.2 Single-period liabilities

For the sake of intuition, let us first describe the determination of the required solvency capital under the Cost of Capital approach in a given year t, in a setting where the last payment occurs at the end of year t, so that $L_{t+1} = 0$. Then the market value margin (MVM_t is defined according to (3.7) as the cost of capital of $CoC\%$ in excess of the risk-free rate, charged on the required solvency capital SCR_t (there are no future $SCR_{t+\tau}$-s).

Equations (3.4)–(3.6), together with $L_{t+1} = 0$, imply:

$$\begin{aligned} L_t &= BEL_t + CoC \cdot \left(\mathbb{Q}_{1-\alpha,t} \left[\frac{\widetilde{L}_t}{1 + r^{rf}} \right] - L_t \right) \Leftrightarrow \\ L_t &= \left(\frac{1}{1 + CoC} \right) \cdot BEL_t + \left(\frac{CoC}{1 + CoC} \right) \cdot \mathbb{Q}_{1-\alpha,t} \left[\frac{\widetilde{L}_t}{1 + r^{rf}} \right]. \end{aligned} \tag{3.8}$$

Therefore, the required solvency capital SCR_t is given by:

$$\begin{aligned} SCR_t = \; & \mathbb{Q}_{1-\alpha,t} \left[\frac{\widetilde{L}_t}{1 + r^{rf}} \right] - L_t, \\ = & \left(\frac{1}{1 + CoC} \right) \cdot \left(\mathbb{Q}_{1-\alpha,t} \left[\frac{\widetilde{L}_t}{1 + r^{rf}} \right] - BEL_t \right). \end{aligned} \tag{3.9}$$

Thus, in a single-period setting, we obtain closed form expressions for the value of the liabilities, and the required solvency capital.

3.2.3 Multi-period liabilities

In the previous subsection we determined the value of the liabilities and the corresponding capital reserve in the Cost of Capital approach for liabilities with only a single payout. However, the run-off time for liabilities of life insurance products is generally much longer. The intention of the regulator is that the market value margin equals the cost (i.e., $CoC \times 100\%$) charged to the present value of the current and all future values of SCR_{t+s} as given in equation (3.7). However, a problem with this calculation is that the SCR_{t+s} depends on the distribution of the all SCR-s after time $t+s$. To solve this problem, the liabilities are treated as if the run-off period has length one, and the value of the liabilities at the end of time t is given by $\widetilde{L}_t + L_{t+1}$. The interpretation is that the insurer can sell the liabilities at the end of the year at a price equal to L_{t+1}. This effectively transforms the problem to a single-period problem as described in the previous subsection. Specifically, let $BEL_t^{1\text{ period}}$ denote the current best estimate value of the liabilities, given that at the end of year t they will be sold at price L_{t+1}, i.e.,

$$BEL_t^{1\text{ period}} = \mathbb{E}_t\left[\frac{\widetilde{L}_t + L_{t+1}}{1+r^{rf}}\right]. \tag{3.10}$$

Then it follows from (3.8), and $BEL_t^{1\text{ period}}$ defined in this way, that the current market value of the liabilities is given by:

$$\begin{aligned} L_t =& BEL_t^{1\text{ period}} + CoC \cdot SCR_t \\ =& \frac{1}{1+CoC} \cdot \mathbb{E}_t\left[\frac{\widetilde{L}_t + L_{t+1}}{1+r^{rf}}\right] \\ &+ \frac{CoC}{1+CoC} \cdot \mathbb{Q}_{1-\alpha,t}\left[\frac{\widetilde{L}_t + L_{t+1}}{1+r^{rf}}\right]. \end{aligned} \tag{3.11}$$

Even though only the capital charge for the first period appears explicitly in the expression for the value of the liabilities, capital charges for holding the risk in later years are included through their effect on the distribution of the market value of the liabilities next year. Indeed, recursive evaluation of the first equality of equation

(3.11) yields:

$$L_t = BEL_t + MVM_t = BEL_t + CoC \cdot \sum_{s \geq 0} \frac{\mathbb{E}_t \left[SCR_{t+s} \right]}{\left(1 + r^{rf}\right)^s}, \tag{3.12}$$

where

$$SCR_{t+s} = \mathbb{Q}_{1-\alpha,t+s} \left[\frac{\widetilde{L}_{t+s} + L_{t+s+1}}{1 + r^{rf}} \right] - L_{t+s}. \tag{3.13}$$

Thus, the value of the liabilities equals the current best estimate, plus a market value margin MVM_t that equals the present value of the *expected* cost of capital, $\mathbb{E}_t \left[SCR_{t+s} \right]$, associated with holding the liability in future periods.

To determine sufficiently accurately the value of MVM_t in the CoC approach using a simulation approach might require many simulations. The reason is that we have to determine:

$$\mathbb{E}_t \left[SCR_{t+s} \right] = \mathbb{E}_t \left[\mathbb{Q}_{1-\alpha,t+s} \left[\frac{\widetilde{L}_{t+s} + L_{t+s+1}}{1 + r^{rf}} \right] - L_{t+s} \right], \tag{3.14}$$

where the future SCR_{t+s} itself depends on the conditional expectation of SCR_{t+s+s_1}-s even further in the future. Solvency II allows to make simplifications when these would not lead to a result which is materially different from the result which would result from a more accurate valuation process.[3] We consider these in the next section. Another approach to determine the value of the liabilities, without time consuming simulations, is to make use of closed form approximations of the distribution of the variables of interest. We consider this possibility in Section 3.4.

3.3 Simplifications in the CoC-approach

In this section we discuss some simplifications to calculate the capital requirements in the internal model, based on proposals in Solvency II. In these simplification the market values of the liabilities is set equal their best estimate, i.e., at this stage one sets $MVM_{t+s} = 0$, resulting in solvency capital requirements that are known at time t. To indicate this, we denote the simplification of SCR_{t+s} by $SCR_t^{(s)}$. Given

[3] According to section TS.II.C.16 in QIS 4, and section TP.5.32 in QIS 5 the SCR can be calculated using either a direct application of SCR formulae or using the proposed simplifications.(see QIS 4, section TS.II.A.35, QIS 5, section TP.5.30)

these simplifications, we can calculate the corresponding simplification of MVM_t, using (3.7), with SCR_{t+s} replaced by $SCR_t^{(s)}$.

Longevity shock approach. In order to calculate the capital requirements for longevity risk in the Solvency II proposal, QIS 4 (sections TS.XI.C and TS.XII.D.28) and QIS 5 (sections SCR 7.3 and SCR 7.28) proposes a simplified approach and referred to as the *standard model*. In this simplified approach, the required solvency capital in any future period $t+s$, $SCR_t^{(s)}$, is defined as the *change* in the net asset value at time $t+s+1$ due to a (permanent) 20% decrease in mortality probabilities for each age, compared to their current best estimates.[4] The net asset value in year $t+s+1$, i.e., the value of assets minus the value of the liabilities, is given by:

$$A_{t+s+1} - L_{t+s+1} = \left(1 + r^{rf}\right) \cdot A_{t+s} - (\widetilde{L}_{t+s} + L_{t+s+1}).$$

Assuming that r^{rf} and A_{t+s+1} are unaffected by the mortality drop, we have that:

$$SCR_t^{(s)} = \widetilde{L}_{t+s}^{BE} + L_{t+s+1}^{BE} - \left(\widetilde{L}_{t+s}^{SHOCK} + L_{t+s+1}^{SHOCK}\right),$$

where $\widetilde{L}_{t+s}^{BE}$ and L_{t+s+1}^{BE} represent the expected value of payments in year $t+s$ and the market value of the liabilities in the next year, respectively, in a scenario in which future one-year death probabilities are equal to their current best estimate value, and where $\widetilde{L}_{t+s}^{SHOCK}$ and L_{t+s+1}^{SHOCK} represent the expected value of payments in the year $t+s$ and the market value of the liabilities in the next year, respectively, in a scenario in which future death probabilities are equal to 80% of the best estimate value at time t. Because in both scenarios, future death probabilities are taken as given (and are either equal to the current best estimate value or 80% of this value), the corresponding market values at date $t+s+1$ are equal to the present value of all future payments, given the death probabilities in the corresponding scenario. Therefore,

$$\begin{aligned} SCR_t^{(s)} &= \widetilde{L}_{t+s}^{shock} - \widetilde{L}_{t+s}^{BE} + \sum_{s_1 \geq 0} \frac{\widetilde{L}_{t+s+1+s_1}^{SHOCK}}{\left(1 + r^{rf}\right)^{s_1+1}} - \sum_{s_1 \geq 0} \frac{\widetilde{L}_{t+s+1+s_1}^{BE}}{\left(1 + r^{rf}\right)^{s_1+1}} \\ &= \sum_{s_1 \geq 0} \frac{\widetilde{L}_{t+s+s_1}^{SHOCK} - \widetilde{L}_{t+s+s_1}^{BE}}{\left(1 + r^{rf}\right)^{s_1}}. \end{aligned} \tag{3.15}$$

[4]See QIS 5, section SCR 7.28. The permanent decrease of 20% is a revision of the earlier decrease of 25% in QIS 4 (see QIS 4, sections TS.II.A.10 and TS.II.B), which was based on ICAS submission in the UK. The average stress test for longevity risk an insurer in the UK used was 18%, with a range of between 5% and 35% in 2004.

Best estimate scenario. The second approach simplifies equation (3.14) by means of two modifications. First, L_{t+s} and L_{t+s+1} in this equation are replaced by their best estimates BEL_{t+s} and BEL_{t+s+1}, respectively, using (3.6). Next, instead of calculating the expectation over SCR_{t+s}, as defined in (3.13), $SCR_t^{(s)}$ is obtained by evaluating SCR_{t+s} in a single scenario, namely the one in which the future one-year death probabilities up to time $t+s$ evolve according to the time-t best estimate values. Denote this scenario by ω_{t+s}^{BE}. Then

$$SCR_t^{(s)} = SCR_{t+s}\left(\omega_{t+s}^{BE}\right), \tag{3.16}$$

where the right hand sides indicates the realization of the random variable SCR_{t+s} in case of scenario ω_{t+s}^{BE}. We shall refer to this simplification as the BE-simplification.

Fraction of current SCR. As third simplification, we consider for $s \geq 1$

$$SCR_t^{(s)} = SCR_t^{(0)} \times \frac{\mathbb{E}_t\left[BEL_{t+s}\right]}{BEL_t}, \tag{3.17}$$

with $SCR_t^{(0)}$ as defined by (3.16). The idea behind this simplification is that the future SCR as fraction of the best estimate of the liabilities is equal to the current SCR as a fraction of the current best estimate of the liabilities. In case of this simplification, the market value margin is given by:

$$\begin{aligned}
MVM_t =& CoC \cdot \sum_{s\geq 0} \frac{\mathbb{E}_t\left[BEL_{t+s}\right]}{BEL_t} \cdot \frac{SCR_t^{(0)}}{\left(1+r^{rf}\right)^s} \\
=& CoC \cdot \frac{\sum_{s\geq 0} \mathbb{E}_t\left[\sum_{s_1\geq 0} \frac{\widetilde{L}_{t+s+s_1}}{\left(1+r^{rf}\right)^{s+s_1}}\right]}{BEL_t} \cdot SCR_t^{(0)} \\
=& CoC \cdot \frac{\sum_{s\geq 0} (s+1) \frac{\mathbb{E}_t\left[\widetilde{L}_{t+s}\right]}{\left(1+r^{rf}\right)^s}}{\sum_{s\geq 0} \frac{\mathbb{E}_t\left[\widetilde{L}_{t+s}\right]}{\left(1+r^{rf}\right)^s}} \cdot SCR_t^{(0)} \\
=& CoC \cdot Dur_t \cdot SCR_t^{(0)},
\end{aligned} \tag{3.18}$$

where Dur_t denotes the (Macaulay) duration of the best estimate of the liabilities. This simplification is often used, see, for example, the Swiss Solvency Test (SST). See also QIS 4, section TS.II.C.28 and QIS 5 section SCR 7.32.

Simplification (3.18) can easily be adapted by using the modified duration, instead of the Macaulay duration, i.e.,

$$MVM_t = CoC \cdot MD_t \cdot SCR_t, \tag{3.19}$$

where MD_t denotes the modified duration (i.e., the Macaulay duration divided by one plus the yield to maturity). This simplification is proposed in Solvency II, see QIS 4, section TS.II.C.26 and QIS 5 section SCR 7.32. In our case the modified duration is the Macaulay duration divided by one plus the risk free interest rate r^{rf}. Thus, simplification (3.19) follows straightforwardly from simplification (3.18), by dividing the latter by one plus the risk free interest rate. In Section 3.5 we shall present the simplification results in terms of (3.18), and we shall refer to this simplification as the FS-simplification.

3.4 Model for longevity risk

In this section we describe the model yielding the probability distribution $\mathbb{P}_t$ and our approximation $\widetilde{\mathbb{P}}_t$, where the latter will allow us to calculate the capital requirements without having to use simulations. In Subsection 3.4.1 we describe the uncertainty in future forces of mortality, quantified using the Lee and Carter (1992) model. This results in $\mathbb{P}_t$. In Subsection 3.4.2 we describe the life insurance products and our approximation $\widetilde{\mathbb{P}}_t$.

3.4.1 Lee-Carter model

In this section we present our model for $\mathbb{P}_t$. For this purpose we will use the Lee-Carter model, with some additional assumptions. Let $x \in \{0, 1, ..., MA\}$ be the age, where MA is the maximum attainable age (set at 110), let $g \in \{M, F\}$ be the gender, where M is males and F is females, let t be the base year, and let $s \in \{1, 2, 3, \ldots\}$ be the number of years since the base year. The version of the Lee-Carter model that we consider assumes that the age and time dependent one-year force of mortality with age dependent parameters a_x^g and b_x^g, and time dependent parameter k_{t+s}^g, is given by:

$$\mu_{x,t+s}^g = \exp\left(a_x^g + b_x^g k_{t+s}^g + \epsilon_{x,t+s}^g\right), \tag{3.20}$$

where the $\epsilon_{x,t+s}^g$ represent the measurement errors. Introduce

$$\epsilon_{t+s}^g = \left(\epsilon_{1,t+s}^g, \ldots, \epsilon_{MA,t+s}^g\right)', g \in \{M, F\},\ s \in \{1, 2, 3, \ldots\}.$$

Then the Lee-Carter model assumes

$$\epsilon_{t+s} \equiv \begin{pmatrix} \epsilon_{t+s}^M \\ \epsilon_{t+s}^F \end{pmatrix} \overset{i.i.d}{\sim} N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \Sigma_\epsilon^{MM} & \Sigma_\epsilon^{MF} \\ \Sigma_\epsilon^{FM} & \Sigma_\epsilon^{FF} \end{pmatrix}\right). \tag{3.21}$$

Following typical findings in the empirical literature, see, among others, Lee and Carter (1992), Renshaw and Haberman (2006), and Booth, Hyndman, Tickle, and de Jong (2006), we postulate that the evolution over time of k^g_{t+s}, $g \in \{M, F\}$, $s \in \{1, 2, 3, \ldots\}$, can be described by a random walk with drift

$$k^g_{t+s} = k^g_{t+s-1} + c^g + e^g_{t+s}, \tag{3.22}$$

with k^g_t given, and with

$$e_{t+s} \equiv \begin{pmatrix} e^M_{t+s} \\ e^F_{t+s} \end{pmatrix} \overset{i.i.d}{\sim} N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \left(\sigma^M_e\right)^2 & \rho_e \sigma^M_e \sigma^F_e \\ \rho_e \sigma^M_e \sigma^F_e & \left(\sigma^F_e\right)^2 \end{pmatrix} \right), \tag{3.23}$$

independent of ϵ_{t+s}. To avoid identifiability problems, we set $\sum_x b^g_x = 1$, and $k^g_t = 0$. Combining this with equations (3.20) and (3.22) we have, for $s \geq 1$,

$$\log\left(\mu^g_{x,t+s}\right) = a^g_x + s b^g_x c^g + b^g_x \sum_{\tau=1}^{s} e^g_{t+\tau} + \epsilon^g_{x,t+s}. \tag{3.24}$$

We set $\mu^g_{x,t}$ equal to its observation in order to prevent jump-off bias, see Booth, Maindonald, and Smith (2002b). To complete the model description, we allow for uncertainty in the parameters c^F and c^M, i.e., we assume

$$c \equiv \begin{pmatrix} c^M \\ c^F \end{pmatrix} \sim N\left(\begin{pmatrix} \mu^M_c \\ \mu^F_c \end{pmatrix}, \begin{pmatrix} \left(\sigma^M_c\right)^2 & \rho_c \sigma^M_c \sigma^F_c \\ \rho_c \sigma^M_c \sigma^F_c & \left(\sigma^M_c\right)^2 \end{pmatrix} \right), \tag{3.25}$$

independent of ϵ_{t+s} and e_{t+s}. We include parameter uncertainty in the drift terms, c, to take into account uncertainty in the expected trend in future mortality.

Introduce now the index set

$$\mathcal{I} = \{(x, s, g) \mid x \in \mathcal{X}, s \in \mathcal{T}_x, g \in \{M, F\}\}$$

with $x \in \mathcal{X} \equiv \{0, 1, \ldots, MA\}$ representing the age class, $s \in \mathcal{T}_x \equiv \{1, \ldots, MA - x\}$ the time period relative to the base year, and g the gender. Define the vector ℓ_t, containing the change in the log mortality rates over time, with components $\ell_t(i)$, for $i = (x, s, g) \in \mathcal{I}$, where x is the age in the base year t, by,

$$\ell_t(i) = \log\left(\mu^g_{x+s,t+s}\right) - \log\left(\mu^g_{x+s,t}\right). \tag{3.26}$$

Using (3.24) we find

$$\begin{aligned} \ell_t(i) = & \left(a^g_{x+s} + s b^g_{x+s} c^g + b^g_{x+s} \sum_{\tau=1}^{s} e^g_{t+\tau} + \epsilon^g_{x+s,t+s} \right) - \left(a^g_{x+s} + \epsilon^g_{x,t} \right) \\ = & s b^g_{x+s} c^g + b^g_{x+s} \sum_{\tau=1}^{s} e^g_{t+\tau} + \left(\epsilon^g_{x+s,t+s} - \epsilon^g_{x+s,t} \right). \end{aligned}$$

Straightforward calculations result in the following lemma, describing $\mathbb{P}_t$ in terms of these log mortality rates. In this lemma we denote by $\mathcal{F}_t$ the sigma field representing the information up to and including time t.

Lemma 3.4.1

$$\ell_t \mid \mathcal{F}_t \sim N\left(\mu_\ell, \Sigma_\ell\right) \tag{3.27}$$

with μ_ℓ the mean vector with components

$$\mu_\ell\left(i\right) = s b^g_{x+s} \mu^g_c,$$

for $i = (x, s, g) \in \mathcal{I}$, and with Σ_ℓ the covariance matrix, with components

$$\Sigma_\ell\left(i, j\right) = b^g_{x+s} b^h_{y+S} f\left(i, j\right) + 2\Sigma^{g,h}_\epsilon(x+s, y+S),$$

for $i = (x, s, g), j = (y, S, h) \in \mathcal{I}$, where $\Sigma^{g,h}_\epsilon(x+s, y+S)$ is the $(x+s, y+S)$-component of $\Sigma^{g,h}_\epsilon$ defined in Equation (3.21), and where

$$f\left(i, j\right) = \left(1_{(g=h)} + 1_{(g\neq h)}\rho_e\right) \times \sigma^g_e \sigma^h_e \min(s, S) + \left(1_{(g=h)} + 1_{(g\neq h)}\rho_c\right) \times \sigma^g_c \sigma^h_c sS.$$

Using the components of the vector ℓ_t, we introduce the vector of Reduction Factors rf_t, with components $rf_t\left(i\right)$, for $i \in \mathcal{I}$, as follows

$$rf_t\left(i\right) = \exp\left(\ell_t\left(i\right)\right). \tag{3.28}$$

Let $X \sim N\left(\mu, \Sigma\right)$, and $Y = \exp\left(X\right)$. Then (by definition) $Y \sim \log N\left(\mu, \Sigma\right)$. Thus, using Lemma 1, we have $rf_t \mid \mathcal{F}_t \sim \log N\left(\mu_\ell, \Sigma_\ell\right)$. We define the vector $\widetilde{\mu}_t$, with components $\widetilde{\mu}_t\left(i\right)$, for $i = (x, s, g) \in \mathcal{I}$, by

$$\widetilde{\mu}_t\left(i\right) = rf_t\left(i\right) \times \widehat{\mu}^g_{x+s,t}, \tag{3.29}$$

with $\widehat{\mu}^g_{x+s,t}$ the actually observed value of $\mu^g_{x+s,t}$, used as "starting value" to avoid a jump-off bias. We define the vector $\widetilde{\ell}_t$, with components $\widetilde{\ell}_t\left(i\right)$, for $i = (x, s, g) \in \mathcal{I}$, by $\widetilde{\ell}_t\left(i\right) = \log\left(\widehat{\mu}^g_{x+s,t}\right)$. Then we have

$$\widetilde{\mu}_t \mid \mathcal{F}_t \sim \log N\left(\widetilde{\ell}_t + \mu_\ell, \Sigma_\ell\right). \tag{3.30}$$

Given $\widetilde{\mu}_t$, we can define the survival probabilities. For instance, for $i = (x, s, g) \in \mathcal{I}$ the realized one year survival probability of an individual with gender g and age $x+s$ at time $t+s$ is given by $p^g_{x+s,t+s} = \exp\left(-\widetilde{\mu}\left(x, s, g\right)\right)$, while the realized s year survival probability for an individual of gender g and age x at time t is then given by ${}_s p_{x,t} = \prod_{\tau=1}^s p_{x+\tau,t+\tau}$.

3.4.2 Approximation

The distribution $\mathbb{P}_t$ in terms of log mortality rates is rather simple. However, we need the far more complicated induced probability distribution of the value of the liabilities L_{t+s}, $s \geq 0$, as defined by backward recursion by equation (3.11). In this subsection we present the time-specific liability payments $\widetilde{L}_{t+s}$, $s \geq 0$, that we investigate in this chapter, and our approximation for their (induced) probability distribution. Then we discuss our approximation for the (induced) probability distribution for the value of the liabilities L_{t+s}.

We consider the following two types of life insurance products:

1. A *single life annuity* consisting of a nominal yearly payment of 1 with a first payment in the year in which the participant reaches the age of 65, with a last payment in the year (s)he dies;

2. A *survivor annuity* consisting of a nominal yearly payment of 1 with a first payment in the year in which the participant dies, with a last payment in the year his (her) partner dies.

Introduce the index set

$$\begin{aligned}\mathcal{J} = \big\{&(a, x, g, p, y_p, g_p, s) \mid a \in \mathcal{A}, x, y_p \in \mathcal{X}_0 \subset \mathcal{X}, s \in \mathcal{T}_{\min\{x,y_p\}}\\ &g, g_p \in \{M, F\}, p \in \{0, 1\}\big\},\end{aligned}$$

with a representing the life insurance product from the product class $\mathcal{A}$, with $p = 1$ in case a partner is present, and $p = 0$ otherwise, with x representing the age class of the insured, with y_p the age class of the partner of the insured, where $\mathcal{X}_0 \subset \mathcal{X}$ represents the set of ages of the insureds included in the portfolio under consideration, and with g_p representing the gender of the partner, where we set $y_0 = x$ and $g_0 = g$ in case the insured does not have a partner ($p = 0$).

In this chapter we consider four portfolios, which are defined in Section 3.5. Since these portfolios consist of single life annuities ($a = sl$) and survivor annuities ($a = sa$), we have $A = \{sl, sa\}$. Moreover, we shall consider portfolios with insureds (and their possible partners) having age 65 at time t, so that $\mathcal{X}_0 = \{65\}$.

We introduce the vector $\widetilde{L}$, containing the possible time- and product specific liability payments, with components $\widetilde{L}\,(j)$, $j \in \mathcal{J}$, where each $\widetilde{L}\,(j)$ represents the liability payment corresponding to product $j \in \mathcal{J}$. We have

$$\widetilde{L}\,(sl, x, g, p, y_p, g_p, s) = 1_{[r_j,\infty)}(s) \times {}_sp^g_{x,t}$$

for a single-life annuity at time $t+s$, in case of an insured of gender g and age x at the base year t, where $r_j = \max\{65-x,1\}$, the number of years until the first single life annuity payment, and[5]

$$\widetilde{L}\left(sa,x,g,p,y_p,g_p,s\right) = p \times \left(1 - {}_sp^{g}_{x,t}\right) \times {}_sp^{g_p}_{y_p,t}$$

for a survivor annuity at time $t+s$, in case of an insured of gender g and age x at the base year t, and a partner of gender g_p and age y_p, also at the base year t.

In the appendix we derive as approximation for the distribution of the vector $\widetilde{L}$, conditional on the information set $\mathcal{F}_t$, a log normal distribution, with appropriately chosen mean vector and covariance matrix, starting from the distribution of $\widetilde{\mu}_t \mid \mathcal{F}_t$, see (3.30). We motivate this approximation as follows. First, we approximate the lognormal distribution of the mortality rates by a normal distribution, where the parameters are set such that the first two moments of the two distributions match. We show that this approximation is rather close, also because longevity risk, although systematic, is not too large a risk factor. Next, we apply an extension of the Fenton-Wilkinson approximation for a linear transformation of lognormally distributed random variables to arrive at the approximate probability distribution of $\widetilde{L}$. This approximation is also quite accurate, again because longevity risk is not too large a risk factor.[6] For a more detailed description of the approximations and a quantification of the accuracy of the approximations, we refer to Appendix 3.A.1.[7]

Given the approximate distribution of the vector $\widetilde{L}$, containing the time- and product-specific liability payments, our approximation of the distribution of the time-specific liability payments $\widetilde{L}_{t+s}$ follows by an application of the Fenton-

[5]Existing literature shows that there exists dependence between the remaining lifetimes of a participant and his (her) partner at micro-level, e.g., due to the fact that partners have similar lifestyles, or that the passing away of a partner affects the surviving relative's quality of life. Because our focus in this chapter is on systematic longevity risk, we ignore this dependence and assume that the remaining lifetimes of the spouses, conditional on the survival probabilities, are independent.

[6]Indeed, in the limit, when the variance goes to zero, one can show that a *sum* of lognormal distributed variables is lognormally distributed, see Dufresne (2004). We illustrate that we also have a good approximation in case of the linear transformations relevant for our approximation.

[7]Our approach to obtain the distribution of the future cash flows of life insurance products is different from the comonotonic quantile-additivity approach developed in Denuit and Dhaene (2007) and Denuit (2008), since it does not require the assumption of comonotonic random variables (i.e., the mortality rates) and monotonic function of these variables. Moreover, in our model we do not need the restriction that all the age specific parameters β_x should have the same sign, which may not be the case, see, for example, Brouhns, Denuit, and Vermunt (2002a), where the sign of the β_x switch from positive to negative at the age of 95 for males and 97 for females in case of Belgian mortality data.

Wilkinson approximation for sums of lognormal distributions, since the time-specific liability payment $\widetilde{L}_{t+s}$ is just a linear transformation of the vector $\widetilde{L}$ of time- and product-specific liability payments.

Next, the distribution of the value of the liabilities L_{t+s} can be obtained by applying (3.11) in a backward recursion, starting from some $L_{t+\tau}$. In case of a run-off approach, as we consider in Section 3.5, we can take $\tau = T+1$ for which we will have $L_{t+T+1} = 0$. Our approximation is such that the time-specific liability payments $\widetilde{L}_{t+T}$ *simultaneously* with the vector containing the realized survival probabilities is (approximately) lognormally distributed. Motivated by the Markov-property of these realized survival probabilities, in case of the Lee-Carter model, we assume that $\mathcal{F}_{t+T}$ can be approximated by the sigma field generated by the realized survival probabilities corresponding to period $t+T$. Using the characteristics of a lognormal distribution, we are then able to calculate the recursion (3.11) for the time step from $T+1$ to T, yielding as approximation for L_{t+T} a lognormal distribution, with appropriate mean vector and covariance matrix. The Fenton-Wilkinson approximation then yields that the sum of L_{t+T} and $\widetilde{L}_{t+T-1}$ is (approximately) lognormal. Using backward induction we are then able to obtain in a similar way approximate lognormal distributions for the earlier L_{t+s}, $s \leq T-1$. For a more detailed description we refer to Appendix 3.A.1. In this appendix we also compare the aggregated discounted differences in the best estimate scenarios between the SCR, using $\mathbb{P}_t$-based simulations of the mortality probabilities and using our approximation. We find for different portfolios of life insurance products that the aggregated discounted differences are small, up to approximately 1.8%. The discounted sum of SCR-s, multiplied with the cost of capital rate, gives MVM_t, hence, the results indicate that the error using the model for calculating the market value of liabilities is small.

3.5 Solvency capital requirements

In this section we consider capital requirements for a portfolio of life insurance products and compare the results to the simplified approach in the Solvency II proposal. We will consider the following four portfolios:

1: 100% male, aged 65, with a single life annuity;

2: 100% female, aged 65, with a single life annuity;

3: 50% male, aged 65, with a single life annuity and 50% female, aged 65, with a single life annuity;

4: 50% male, aged 65, with a single life annuity and a survivor annuity, and 50% female, aged 65, with a single life annuity and a survivor annuity. The partner is of the opposite gender with age 65 and the survivor annuity payments are 70% of the single life annuity payments.

We assume that the risk free one year return is equal to 4%, resulting in $P_t^{(s)} = \frac{1}{1.04^s}$ for all t and s. To estimate the parameters of the distribution of the future mortality rates we use US, UK, and Dutch age and gender specific mortality data from 1970 to 2006, representing a large, a medium, and a small population size, respectively.[8] A detailed description of the method to estimate the parameters of the Lee-Carter model and the resulting parameter estimates are presented in Appendix 3.A.2.

In Subsection 3.5.1 we first report the capital requirements resulting from the closed form approximation in the internal model. In Subsection 3.5.2 we consider the capital requirements using the Solvency II simplifications and the simplifications to the internal model as described in Section 3.3 and we discuss the differences between the approximation and the simplifications.

3.5.1 The capital requirements using the approximation

In this subsection we present the capital requirement for the US, UK, and the Netherlands for four different portfolios of 65 years old insureds, described in the previous section, using our approximation, as described in the previous section. In QIS 4 and 5 the CoC-rate is set at 6%, which is interpreted as the cost of capital in excess of the risk free interest rate. To deal with the reaction of the industry to the Solvency proposal that this CoC-percentage is set too high, we also use the CoC-percentage of 4%, instead of only the proposed 6%.[9] The results are displayed in Table 3.5.1.

[8] We use a relatively short time period for the estimation of the parameters, because otherwise the assumption of time-invariant parameter b_x might not be valid, see Tuljapurkar et al. (2000), Lee and Miller (2001), and Booth et al. (2002b).

[9] See the FSA UK country report (2008) and CEA (the European insurance and reinsurance federation) (2009).

Table 3.5.1: **Table with capital requirements using the internal model**

	$\frac{A_t^\star \text{ - BEL}_t}{\text{BEL}_t}$	$\frac{\text{MVM}_t}{\text{BEL}_t}$	$\frac{\text{SCR}_t}{\text{BEL}_t}$	$\frac{A_t^\star \text{ - BEL}_t}{\text{BEL}_t}$	$\frac{\text{MVM}_t}{\text{BEL}_t}$	$\frac{\text{SCR}_t}{\text{BEL}_t}$
	US, CoC=6%			US, CoC=4%		
1	1.47%	1.04%	0.44%	1.16%	0.71%	0.45%
2	1.17%	0.77%	0.41%	0.93%	0.52%	0.41%
3	1.17%	0.80%	0.38%	0.92%	0.55%	0.38%
4	0.97%	0.74%	0.24%	0.74%	0.51%	0.24%
	UK, CoC=6%			UK, CoC=4%		
1	2.59%	1.73%	0.89%	2.06%	1.17%	0.91%
2	2.47%	2.04%	0.47%	1.87%	1.42%	0.47%
3	2.22%	1.70%	0.55%	1.70%	1.16%	0.55%
4	2.00%	1.63%	0.39%	1.49%	1.11%	0.39%
	NL, CoC=6%			NL, CoC=4%		
1	1.74%	0.97%	0.78%	1.45%	0.66%	0.80%
2	2.92%	2.38%	0.60%	2.19%	1.60%	0.61%
3	2.01%	1.54%	0.50%	1.53%	1.04%	0.50%
4	1.82%	1.50%	0.34%	1.34%	1.00%	0.35%

This table displays MVM and SCR in the internal model, as percentage of the best estimate of the liabilities, for four different portfolios of life insurance products for the US, UK, and the Netherlands. The market value of the liabilities is set according the internal model with a Cost of Capital rate of 6% in columns 2 till 4 and a Cost of Capital rate of 4% in columns 5 till 7.

As expected, we observe that the MVM_t, using a Cost of Capital rate of 4%, is lower than the MVM_t using a Cost of Capital rate of 6%. We also observe that the capital reserves ($A_t^\star$, MVM_t, and SCR_t) for the different portfolios depend significantly on the underlying population. For example, the capital reserves in the UK are higher than in the US for all four investigated portfolios. Moreover, a single life annuity for males compared to a single life annuity for females has a lower capital requirement in the Netherlands, but a higher capital requirement in the US and UK. Hence, a simple rules of thumb for all underlying populations by setting the MVM_t and SCR_t for an insured aged 65 equal to a given percentage might not be accurate for valuing the liabilities of all investigated populations.

3.5.2 The capital requirements using simplifications

In this section we report the capital requirements for the four portfolios corresponding to the simplifications discussed in Section 3.3. We first consider the *standard* *model* with the 20% longevity shock. Table 3.5.2 displays the resulting capital reserves for the four portfolios, i.e., the market value margin (MVM_t), the solvency capital requirement ($SCR_t^{(0)}$),[10] and the total capital requirement ($A_t^\star$).

Table 3.5.2: **Table with capital requirements using the longevity shock approach**

	Longevity shock approach					
	$\frac{A_t^\star - \text{BEL}_t}{\text{BEL}_t}$	$\frac{\text{MVM}_t}{\text{BEL}_t}$	$\frac{\text{SCR}_t}{\text{BEL}_t}$	$\frac{A_t^\star - \text{BEL}_t}{\text{BEL}_t}$	$\frac{\text{MVM}_t}{\text{BEL}_t}$	$\frac{\text{SCR}_t}{\text{BEL}_t}$
	CoC = 6%			CoC = 4%		
	US					
1	15.98%	8.23%	7.75%	13.24%	5.49%	7.75%
2	13.35%	7.21%	6.14%	10.95%	4.81%	6.14%
3	14.59%	7.69%	6.90%	12.02%	5.13%	6.90%
4	12.87%	7.27%	5.60%	10.45%	4.85%	5.60%
	UK					
1	16.15%	8.26%	7.89%	13.40%	5.51%	7.89%
2	13.34%	7.20%	6.14%	10.94%	4.80%	6.14%
3	14.65%	7.69%	6.96%	12.09%	5.13%	6.96%
4	13.14%	7.39%	5.75%	10.67%	4.93%	5.75%
	NL					
1	15.72%	7.88%	7.83%	13.09%	5.26%	7.83%
2	12.24%	6.65%	5.59%	10.02%	4.43%	5.59%
3	13.81%	7.21%	6.61%	11.41%	4.80%	6.61%
4	12.37%	6.93%	5.44%	10.06%	4.62%	5.44%

This table displays the capital reserves (MVM_t and SCR_t), as percentage of the best estimate of the liabilities (BEL_t), for four different portfolios of life insurance products for the US, UK, and the Netherlands for $t = 2006$. The market value of the liabilities is set according to the longevity shock approach proposed in Solvency II, using a Cost of Capital rate of 6% and 4%.

The capital requirements in the standard model are significantly larger than in case of the internal model, using our approximation. According to the simplified approach of the standard model, for an insured aged 65 years, depending on the portfolio composition and the CoC percentage, an insurer should hold between 10% and just more than 16% of the best estimate value of the liabilities in excess of the

[10] This $SCR_t^{(0)}$ is independent of the CoC-rate. This is due to the definition of the SCR in Solvency II. The current SCR is only affected by a change in the best estimate value of the liabilities and not by the market value of the liabilities. This is different from the internal model, where SCR_t does depend on the CoC-rate (see Table 3.5.2)

BEL_t in order to fulfill the capital requirements in the Solvency II proposal, while this is at most around 3% in case of the internal model.

The capital requirements using the simplified approach in the Solvency II standard model depend on the size of the the *longevity shock*. In the document *UNESPA longevity risk investigation* (2009) of the the European insurance and reinsurance federation (CEA), the standard deviations of the annualized five years mortality factors for European mortality probabilities since 1956 are calculated. These mortality factors for age group x in year t, $AMF_{x,t}$ are given by:[11]

$$AMF_{x,t} = \sqrt[5]{\frac{q_{x,t+5}}{q_{x,t}}} - 1, \tag{3.31}$$

where $q_{x,t}$ is the time-t and age-x (population) one year mortality probability, obtained form the Human Mortality Database. The resulting standard deviations turn out to be 1.32%, 1.18%, and 1.01% for the age bands [60–70], [70–80], and [80–90], respectively. Comparing these numbers to the size of the longevity shock equal to 20% suggests that the longevity shock might be quite conservative.

However, whereas these standard deviations are useful to compare the longevity shock in the standard model of Solvency II with the observed shocks in the past, they might not capture possible reductions in future survival probabilities. Especially for contracts with a long maturity, the effect of a change in the trend of future mortality might be substantial.

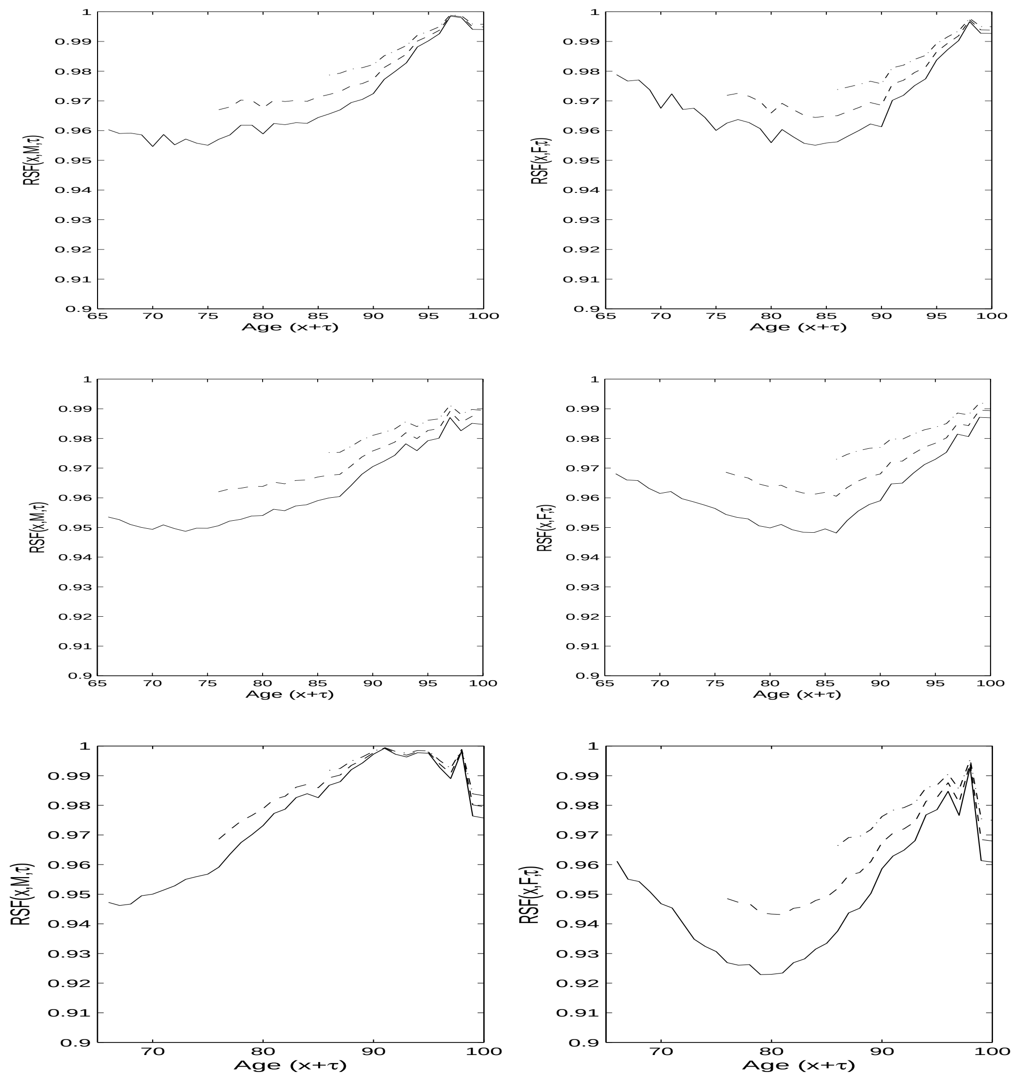
To indicate how large the shocks to the mortality probabilities are in the internal model, we determined the 99.5% quantiles of the expected future survival probabilities within a year. Figure 3.5.1 displays the relative shock factor in mortality probabilities in the 99.5% scenario relative to the best estimate scenario for an individual with age 65 years (solid curve), 75 years (dashed curve), and 85 years (dashed-dotted curve). More precisely, Figure 3.5.1 displays:

$$RSF_t(x,g,s) = \frac{\mathbb{Q}_{t,0.005}\left[q^{g,BE(t+1)}_{x+s,t+s}\right]}{q^{g,BE(t)}_{x+s,t+s}}$$

where $\mathbb{Q}_{t,0.005}\left[q^{g,BE(t+1)}_{x+s,t+s}\right]$ is the time-t 0.5% quantile of the

[11] These annualized 5 year improvement factor are used instead of the annual mortality improvement factors in order to smooth the mortality improvement factors over time, reducing the effect of idiosyncratic longevity risk.

Figure 3.5.1: **Longevity shock scenario in the Lee-Carter model**



The figure displays $RSF_t(x, g, \tau)$ for $x = 65, 75$, and 85 and for $\tau = 1, \ldots, 100-x$, i.e., the shock in mortality probabilities which occurs in the 99.5% percentile of the mortality table within a year for base year $t = 2006$. The upper panels display the shock for the US, the middle panels for the UK, and the lower panels for the Netherlands, the left panels for males (i.e., $g = M$) and the right panels for females (i.e., $g = F$). The solid lines correspond with an individual currently aged 65 (i.e., $x = 65$), the dashed lines correspond with an individual currently aged 75 (i.e., $x = 75$), and the dashed-dotted lines correspond with an individual currently aged 85 (i.e., $x = 85$).

time-$(t+1)$ best estimate of the mortality probability, with x (=65, 75, or 85) the age of the insured at the base year t, $g \in \{M, F\}$ the gender of the insured, and $x+s$ the age to which the relative shock factor applies. The base year is $t = 2006$. Solvency II assumes that $RSF_t(g, x, s) = 0.80$ for $g = M, F$, all x and $s > 0$. Figure 3.5.1 illustrates that the shock in the Lee-Carter model is much smaller than the shock in the simplified approach standard model of Solvency II, but generally of the same magnitude as the corresponding quantile of $AMF_{x,t}$, using a normal distribution with corresponding standard deviation as calculated in *UNESPA longevity risk investigation* (2009). We also observe that, for example, $RSF_t(65, g, 11)$ is approximately 30% higher than $RSF_t(75, g, 1)$. Thus, for contracts with a longer duration the difference between the internal model (using our approximation) and the Solvency standard model becomes smaller.

Next, we present in Table 3.5.3 the market value margin (MVM_t) and the current buffer ($SCR_t^{(0)}$), for the different portfolios, using the other simplifications applied to the internal model, see Table 3.5.3. The two simplifications are the BE-simplification, as given in equation (3.16), and the FS-simplification, as given in equation (3.18). For the sake of comparison, we also include the MVM_t from Table 3.5.1, calculated using our approximation.

From Table 3.5.3 we observe that the BE- and the FS-simplification typically underestimate the risk margin, when the internal model is taken as reference, with only exception the US females insureds in case of the BE-simplification. When comparing $SCR_t^{(0)}$ from Table 3.5.3 and SCR_t from Table 3.5.1, we observe that $SCR_t^{(0)}$ is higher. Nevertheless, MVM_t relative to BEL_t is larger in the internal model than in case of the two simplifications. This is possible, since the MVM_t is the expected discounted value of all future SCR-s multiplied with a Cost of Capital rate, where the expected values of the future SCR-s further away are larger in the internal model than in the simplification.

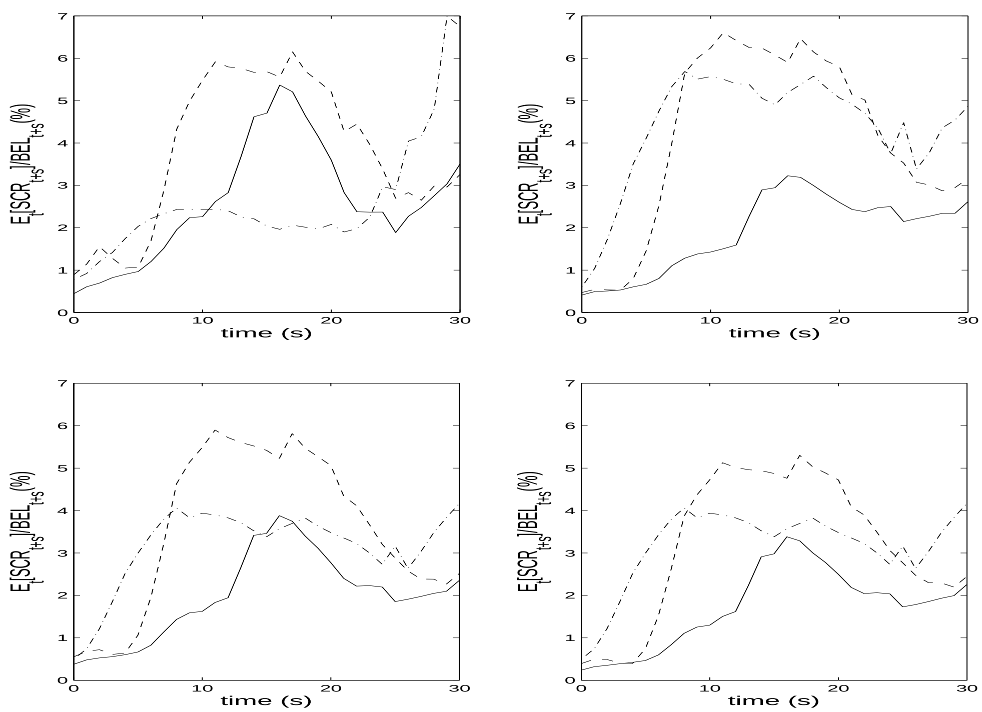
Table 3.5.3: **Table with market value margins and SCR using simplifications**

	$\frac{SCR_t^{(0)}}{BEL_t}$	$\frac{MVM_t}{BEL_t}$					
		internal	BE	FS	internal	BE	FS
	US	US, CoC = 6%			US, CoC = 4%		
1	1.25%	1.04%	0.85%	0.57%	0.71%	0.56%	0.38%
2	0.97%	0.77%	0.80%	0.54%	0.52%	0.46%	0.31%
3	1.01%	0.80%	0.72%	0.48%	0.55%	0.47%	0.31%
4	0.81%	0.74%	0.69%	0.46%	0.51%	0.40%	0.27%
	UK	UK, CoC = 6%			UK, CoC = 4%		
1	1.62%	1.73%	1.06%	0.70%	1.17%	0.73%	0.49%
2	1.50%	2.04%	1.16%	0.77%	1.42%	0.71%	0.48%
3	1.41%	1.70%	0.97%	0.64%	1.16%	0.65%	0.43%
4	1.13%	1.63%	0.92%	0.61%	1.11%	0.56%	0.37%
	NL	NL, CoC = 6%			NL, CoC = 4%		
1	1.51%	0.97%	0.95%	0.63%	0.66%	0.65%	0.43%
2	2.11%	2.38%	1.52%	1.01%	1.60%	1.00%	0.67%
3	1.48%	1.54%	0.99%	0.66%	1.04%	0.67%	0.45%
4	1.28%	1.50%	1.00%	0.67%	1.00%	0.62%	0.41%

This table displays the current required buffer (SCR_t) and the market value margin (MVM_t) in the internal model and using the BE- and FS-simplifications, as percentage of the best estimate of the liabilities (BEL_t), for four different portfolios of life insurance products for the US, UK, and the Netherlands for $t = 2006$. The second column refers to the SCR as given in (3.16), for $s = 0$, which is independent of the Cost of Capital percentage. The market value of the liabilities is set according the internal model with a Cost of Capital percentage of 6% in the third till fifth column and with a Cost of Capital percentage of 4% in the sixth till eighth column. The third and sixth columns refer to the market value margin in the internal model using our approximation, the fourth and the seventh column refer to the BE-simplification as given in equation (3.16), and the fifth and eighth column refer to the FS-simplification as given in equation (3.18).

In the FE-simplification (as given in equation (3.18)) MVM_t is calculated using a constant fraction of SCR-s relative to BEL over the run-off period. A good approximation would require that the SCR relative to BEL is indeed constant over the run-off period. Figure 3.5.2 displays the base year expected SCR in the internal model, i.e., $\mathbb{E}_t\left(SCR_{t+s}\right)$, relative to the base year best estimate of the liabilities over the run-off period.[12] This figure shows that the buffer relative to the best estimate of the liabilities typically first increases when time elapses, and then decreases. The size of the buffer relative to the best estimate in the run-off period is neither constant, nor monotonically increasing, or decreasing, which does not seem to support the idea underlying this simplification.

Figure 3.5.2: **Expected solvency capital requirement in the internal model** as percentage of best estimate of the liabilities in the run-off period



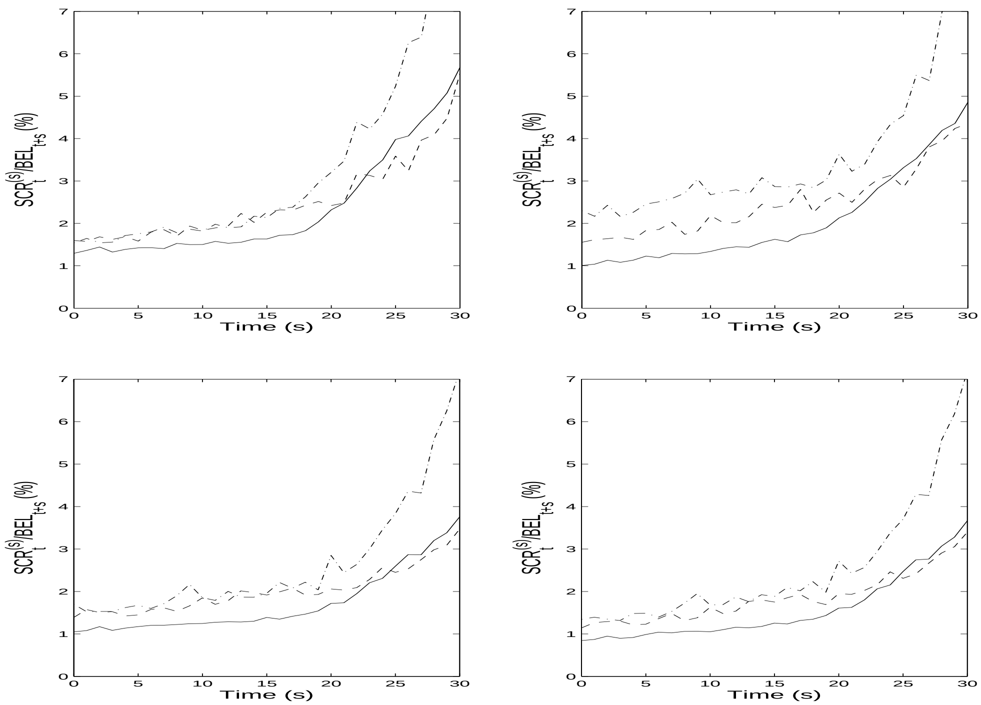
The figure displays the SCR as percentage of the best estimate of the liabilities over time for the run-off period for the US (solid curves), UK (dashed curves), and the Netherlands (dashed-dotted curves). The upper left panel displays the SCR as percentage of the best estimate of the liabilities for portfolio 1, the upper right panel for portfolio 2, the lower left panel displays for portfolio 3, and the lower right panel for portfolio 4.

Figure 3.5.3 displays the solvency capital requirements $SCR_t^{(s)}$ in the BE-simplification as percentage of best estimate of the liabilities in the run-off period.

[12]The curves are not smooth, since we have not smoothed the mortality rates, nor the reduction factor, i.e., we did not smooth b_x in the Lee-Carter model.

Comparing this figure with Figure 3.5.2 we observe that $\mathbb{E}_t\left(SCR_{t+s}\right)$ at intermediate times (for s is approximately 10 up to 25) are underestimated using the BE-simplification, relative to the internal model. In practice, the BE-simplification is often taken to represent the "exact" MVM_t, see Börger (2010). However, our results indicate that this representation might not always be appropriate.

Figure 3.5.3: **Expected solvency capital requirement in the BE simplification** as percentage of best estimate of the liabilities in the run-off period



The figure displays the SCR in the BE simplification as percentage of the best estimate of the liabilities over time for the run-off period for the US (solid curves), UK (dashed curves), and the Netherlands (dashed-dotted curves). The upper left panel displays the SCR as percentage of the best estimate of the liabilities for portfolio 1, the upper right panel for portfolio 2, the lower left panel displays for portfolio 3, and the lower right panel for portfolio 4.

Finally, we compare the resulting initial capital requirements A_t^* (relative to BEL_t), which can be calculated by adding 1, $SCR_t^{(0)}/BEL_t$, and MVM_t/BEL_t (with MVM_t calculated according to the corresponding simplification). Relative to the internal model, we find that A_t^*/BEL_t, calculated according to the BE- or

FS-simplification is still somewhat conservative, but far less so than the standard model (based on the longevity shock). Thus, in terms of initial capital requirements these simplifications seem to work fine.

3.6 Conclusions

This chapter investigates the capital requirements for a portfolio of life insurance products. The capital requirements are set according to the cost of capital approach, in line with the QIS 5 Technical Specification of the Solvency II proposal. The cost of capital approach values non-tradeable risk, such as systematic longevity risk, using an easy to understand method.

However, the cost of capital approach for life insurance products, with typically a long time to maturity, might be computationally demanding when starting from the distribution of mortality rates and using simulation based techniques. We derive a closed form approximation of the induced distribution of the discounted cash flows and the market value of the liabilities of a portfolio of life insurance products starting from the Lee and Carter (1992)-model for the log mortality rates.

Using our approximation we calculate the capital requirements for different portfolios of life insurance products and compare them with the capital requirements using the Solvency II proposed standard formula. The portfolios consist of 65-year-old male and female single life annuity and survivor annuity products for insureds in the US, UK, and the Netherlands.

Our results suggest that the capital requirements using the longevity shock approach in the Solvency II standard model might be quite conservative. On the other hand, the two investigated simplifications turn out to be far less conservative in terms of the initial capital requirements than, much closer to the internal model based calculations, using our approximation. However, we do find substantial differences between these simplification based solvency capital requirements (SCR) and market value margin (MVM) and the corresponding SCR and MVM calculated using the internal model.

Although our approximation turns out to be quite accurate, it is based on the Lee and Carter (1992) model which itself is also an approximation. Thus, the risk quantified according to our internal model is – by approximation – just the risk of the Lee-Carter model, thus, excluding, for example, model risk. It might be interesting to investigate whether accurate approximations can be found for other models than the Lee-Carter model. In this way we might also be able to model risk as well.

3.A Approximations for the Distribution of the Cash Flows

3.A.1 Approximations for the Distribution of the Cash Flows

We start from $\widetilde{\mu}_t$ as defined in (3.30). We shall suppress the index t, unless needed for clarification, since the calculations are made for a given, base year t.

First Approximation. We use as first approximation $\widetilde{\mu} \sim N(\mu_\mu, \Sigma_\mu)$, with the vector μ_μ, with components $\mu_\mu(i)$, for $i \in \mathcal{I}$, given by

$$\mu_\mu(i) \equiv \exp\left(\widetilde{\ell}(i) + \mu_\ell(i) + \frac{1}{2}\Sigma_\ell(i,i)\right),$$

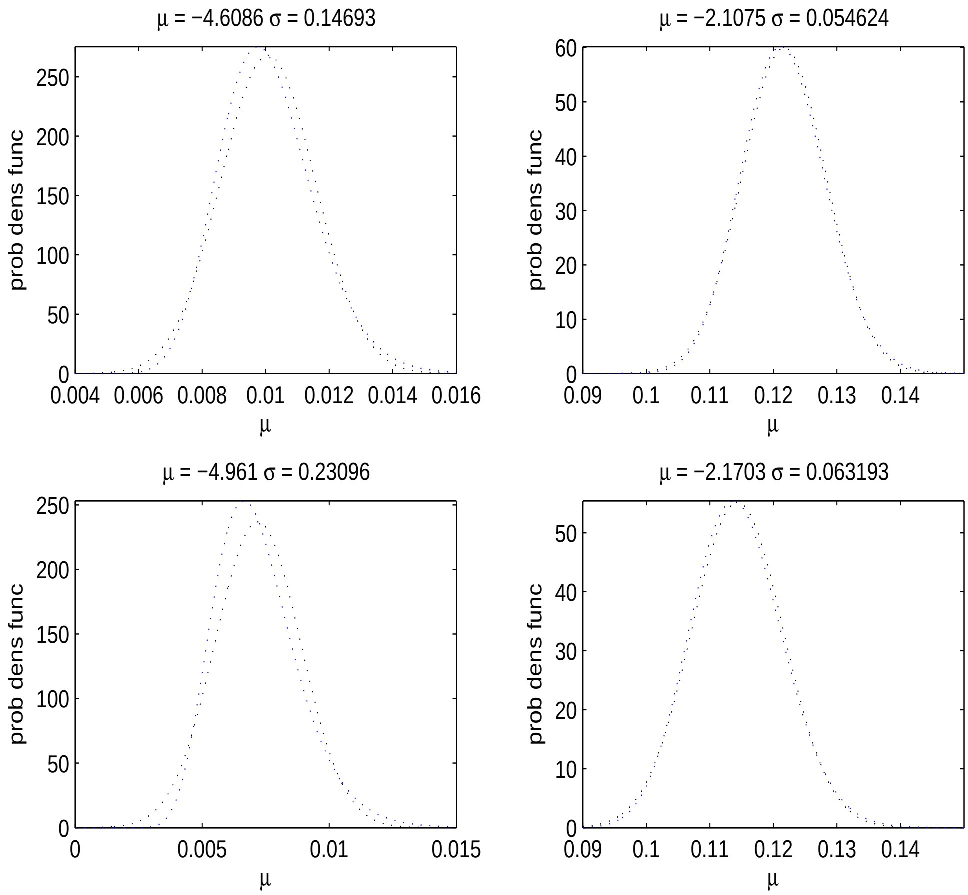
and with the matrix Σ_μ, with components $\Sigma_\mu(i,j)$ for $i = (x,s,g), j = (y,S,h) \in \mathcal{I}$, given by

$$\begin{aligned}\Sigma_\mu(i,j) = &\exp\left(\widetilde{\ell}(i) + \mu_\ell(i) + \widetilde{\ell}(j) + \mu_\ell(j) + \frac{1}{2}\Sigma_\ell(i,i) + \frac{1}{2}\Sigma_\ell(j,j)\right) \\ &\times \left(\exp\left(\Sigma_\ell(i,j)\right) - 1\right).\end{aligned}$$

In this approximation $\mu_\mu(i)$ and $\Sigma_\mu(i,j)$ are set such that they exactly match the corresponding moments of the original distribution.

Accuracy First Approximation. In Figure 3.A.1 and Table 3.A.1 the accuracy of the first approximation is displayed. The upper left panel of Figure 3.A.1 corresponds with the parameters of the distribution of the one-year probability of surviving of a male individual age 100 in 35 years from now ($t = 2006$) in the Netherlands, i.e., $x = 65$, $s = 35$, and $g = M$. The upper right panel corresponds with the parameters of the distribution of the one-year probability of surviving of a male individual individual age 100 in 75 years from now ($t = 2006$), i.e., $x = 25$, $s = 75$, and $g = M$. These long time horizons imply that the uncertainty in the one year survival probabilities is quite large, which would lead to a less accurate approximation than for shorter horizons. The lower left panel displays the effect of an increase in μ for the parameters of the upper right panel. The lower right panel displays the effect of an increase in σ for the parameters of the upper right panel. From the figure and the table we observe that the approximation is accurate. In the tails, and when the variance is high, the approximation is less accurate.

Figure 3.A.1: **Comparison Log Normal - Normal**



This figure displays the probability density function of $X \sim \log N\left(\mu_X, \sigma_X^2\right)$ and the probability density function of the approximation $Y \sim N\left(\mu_Y, \sigma_Y^2\right)$ for different values of $\mu \equiv \mu_X$ and $\sigma \equiv \sigma_X$. The parameters μ_Y and σ_Y^2 are set such that they match the first two moments of those of X.

Table 3.A.1: **Comparison Log Normal-Normal**

	μ_X = -3.9553; σ_X = 0.14693		μ_X = -1.9579; σ_X = 0.054624		μ_X = -4.3077; σ_X = 0.23096		μ_X = -2.0208; σ_X = 0.063193	
Q	X	Y	X	Y	X	Y	X	Y
0.01	0.014	0.013	0.124	0.123	0.008	0.006	0.114	0.113
0.025	0.014	0.014	0.127	0.126	0.009	0.007	0.117	0.116
0.05	0.015	0.015	0.129	0.129	0.009	0.009	0.119	0.119
0.1	0.016	0.016	0.132	0.131	0.010	0.001	0.122	0.122
0.25	0.017	0.017	0.136	0.136	0.012	0.012	0.127	0.127
0.5	0.019	0.019	0.141	0.141	0.013	0.014	0.133	0.133
0.75	0.021	0.021	0.146	0.147	0.016	0.016	0.138	0.138
0.9	0.023	0.023	0.151	0.151	0.018	0.018	0.144	0.144
0.95	0.024	0.024	0.154	0.154	0.020	0.019	0.147	0.147
0.975	0.026	0.025	0.157	0.157	0.021	0.020	0.150	0.149
0.99	0.027	0.026	0.160	0.159	0.023	0.021	0.154	0.152

This table displays the quantiles of the function $X \sim \log N\left(\mu_X, \sigma_X^2\right)$ and the corresponding quantiles of the approximating distribution $Y \sim N\left(\mu_Y, \sigma_Y^2\right)$. The parameters of Y are set such that the first two moments of Y match the first two moments of X. The column with heading Q displays the quantiles. The other columns show the quantiles of X and Y for different values of μ_X and σ_X, where the subcolumns with heading X present the quantiles of the distribution of X, and the subcolumns with heading Y the quantiles of the corresponding distribution of Y.

Fenton-Wilkinson Approximation. All other approximation steps, with one exception, will be based on the Fenton-Wilkinson approximation for sums of lognormal distributions. Suppose A is K-dimensional and G is L-dimensional, simultaneously following the lognormal distribution

$$C \equiv \begin{pmatrix} A \\ G \end{pmatrix} \sim \log N\left(\begin{pmatrix} \mu_{\ell A} \\ \mu_{\ell G} \end{pmatrix}, \begin{pmatrix} \Sigma_{\ell A} & \Sigma_{\ell AG} \\ \Sigma_{\ell GA} & \Sigma_{\ell G} \end{pmatrix}\right) \equiv N\left(\mu_{\ell C}, \Sigma_{\ell C}\right).$$

Let $\mathcal{C}$ denote the index set of the vector C. We have, for $i \in \mathcal{C}$,

$$\mu_C(i) \equiv E(C(i)) = \exp\left(\mu_{\ell C}(i) + \frac{1}{2}\Sigma_{\ell C}(i,i)\right), \tag{3.32}$$

and for $i, j \in \mathcal{C}$,

$$\Sigma_C(i,j) \equiv Cov(C(i), C(j)) = (\exp(\Sigma_{\ell C}(i,j)) - 1) \\ \times \exp\left(\mu_{\ell C}(i) + \mu_{\ell C}(j) + \frac{1}{2}\Sigma_{\ell C}(i,i) + \frac{1}{2}\Sigma_{\ell C}(j,j)\right). \tag{3.33}$$

Next, introduce $B = \Delta A$ for some non-random matrix Δ of dimensions $M \times K$. We define $\mu_B = \Delta\mu_A$, $\Sigma_B = \Delta\Sigma_A\Delta'$, and $\Sigma_{BG} = \Delta\Sigma_{AG} = \Sigma'_{GB}$. Then we take as approximation for the simultaneous distribution of B and G:

$$\begin{pmatrix} B \\ G \end{pmatrix} \sim \log N\left(\begin{pmatrix} \mu_{\ell B} \\ \mu_{\ell G} \end{pmatrix}, \begin{pmatrix} \sigma^2_{\ell B} & \Sigma_{\ell BG} \\ \Sigma_{\ell GB} & \Sigma_{\ell G} \end{pmatrix}\right),$$

with

$$\mu_{\ell B}(j) = \log(\mu_B(j)) - \frac{1}{2}\Sigma_B(j,j), \quad j \in \mathcal{B}$$
$$\Sigma_{\ell B}(j_1, j_2) = \log\left(1 + \frac{\Sigma_B(j_1,j_2)}{\mu_B(j_1)\,\mu_B(j_2)}\right), \quad j_1, j_2 \in \mathcal{B},$$
$$\Sigma_{\ell BG}(j_1, j_2) = \Sigma_{\ell GB}(j_2, j_1) = \log\left(1 + \frac{\Sigma_{GB}(j_1,j_2)}{\mu_B(j_1)\,\mu_G(j_2)}\right), \quad j_1 \in \mathcal{B}, j_2 \in \mathcal{G},$$

where $\mathcal{B} = \{1, \cdots, M\}$ and $\mathcal{G} = \{1, \cdots, K\}$.

For general Δ this approximation is not the Fenton-Wilkinson approximation for *sums* of lognormally distributed random varables. If we apply this approximation to such a case, we shall refer to this approximation as the *generalized* Fenton-Wilkinson approximation, and we shall quantify the corresponding approximation error.

Survival Probabilities. We define the vector $\widetilde{p}$, with components $\widetilde{p}(i)$, for $i = (x, s, g) \in \mathcal{I}$, by

$$\widetilde{p}(i) = \exp(-\widetilde{\mu}(i)). \tag{3.34}$$

By construction, the component $\widetilde{p}(i)$ is the realized one year survival probability of an individual with age $x+s$ and gender g at time-$t+s$ as these are usually calculated in case of the Lee and Carter (1992)-model, including a correction for the jump-off bias. In terms of standard notation, $\widetilde{p}(i)$ is the Lee-Carter estimate of $p^g_{x+s,t+s} = {}_1p^g_{x+s,t+s}$. We define the vector $\widetilde{S}$, with components $\widetilde{S}(i)$, for $i = (x, s, g) \in \mathcal{I}$, by

$$\widetilde{S}(i) = \prod_{\tau=1}^{s} \widetilde{p}(x, \tau, g). \tag{3.35}$$

By construction, the component $\widetilde{S}(i)$ is the realized s year survival probability of an individual with age x at time t and gender g, i.e., in standard notation, $\widetilde{S}(i)$ is the Lee-Carter estimate of ${}_sp^g_{x,t}$. Both $\widetilde{p}$ and $\widetilde{S}$ are random, due to the presence of longevity risk. Straightforward calculations result in the following lemma.

Lemma 3.A.1 *Given the approximation $\widetilde{\mu} \mid \mathcal{F}_t \sim N(\mu_\mu, \Sigma_\mu)$, we have*

$$\begin{pmatrix} \widetilde{p} \\ \widetilde{S} \end{pmatrix} \mid \mathcal{F}_t \sim \log N\left(\begin{pmatrix} \mu_{\ell p} \\ \mu_{\ell S} \end{pmatrix}, \begin{pmatrix} \Sigma_{\ell p} & \Sigma_{\ell pS} \\ \Sigma_{\ell Sp} & \Sigma_{\ell S} \end{pmatrix}\right),$$

with $\mu_{\ell p} = -\mu_\mu$, $\Sigma_{\ell p} = \Sigma_\mu$, and where, for $i = (x, s, g) \in \mathcal{I}$,

$$\mu_{\ell S}(i) = \sum_{\tau=1}^{s} \left(-\mu_\mu(x, \tau, g)\right),$$

and for $i = (x, s, g), j = (y, S, h) \in \mathcal{I}$,

$$\Sigma_{\ell S}(i, j) = \sum_{\tau_1=1}^{s} \sum_{\tau_2=1}^{S} \Sigma_\mu\left((x, \tau_1, g), (y, \tau_2, h)\right),$$

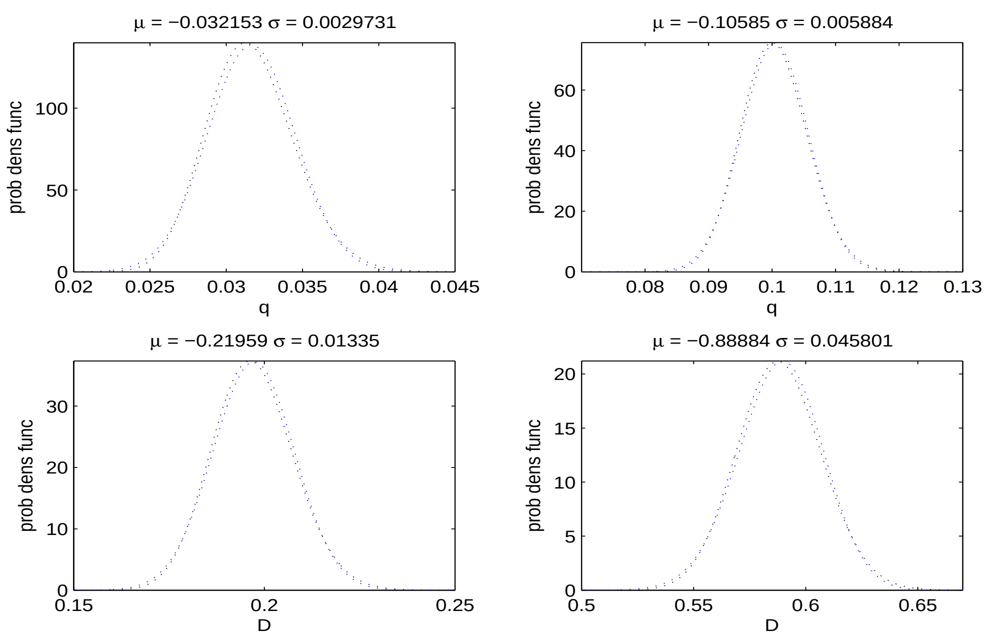
and

$$\Sigma_{\ell Sp}(j, i) = \Sigma_{\ell pS}(i, j) = \sum_{\tau=1}^{S} \Sigma_\mu\left((x, s, g), (y, \tau, h)\right).$$

Second Approximation. Introduce $\widetilde{q} = 1 - \widetilde{p}$ and $\widetilde{D} = 1 - \widetilde{S}$. The component $\widetilde{q}(i) = \widetilde{q}(x, s, g)$ estimates the one-year mortality probabilities of an $x + s$ year old individual in year $t + s$ with gender g, and the component $\widetilde{D}(i) = \widetilde{D}(x, s, g)$ estimates the probability that a currently x year old with gender g will not survive another s years, considered from time t. We define H, with components $H(k, i)$, for $k \in \{1, 2, 3, 4\}$ and $i \in \mathcal{I}$, by $H(1, i) = \widetilde{p}(i)$, $H(2, i) = \widetilde{q}(i)$, $H(3, i) = \widetilde{S}(i)$, and $H(4, i) = \widetilde{D}(i)$. If we take A as a vector with first component 1, and with other components $\widetilde{p}(i)$ and $\widetilde{S}(i)$ for $i \in \mathcal{I}$, then H is a linear transformation of A, and we can apply the generalized Fenton-Wilkinson approximation (without G), if we interpret 1 as a lognormally distributed variable with zero mean and variance (i.e., $1 \sim \log N(0, 0)$), independent of the other components of A. Thus, as our second approximation, we take $H \mid \mathcal{F}_t \sim \log N(\mu_{\ell H}, \Sigma_{\ell H})$, with $\mu_{\ell H}$ and $\Sigma_{\ell H}$ following from the generalized Fenton-Wilkinson approximation.

Accuracy Second Approximation. Figure 3.A.2 and Table 3.A.2 display the accuracy of the second approximation. The upper left panel of Figure 3.A.2 corresponds with the parameters of the distribution of the probability of dying for a Dutch male individual currently aged 65 in 1 year from now. The upper right panel corresponds with the parameters of the distribution of the probability of dying for a Dutch male individual currently aged 65 in 32 year from now. The lower left panel corresponds with the parameters of the distribution of the probability of dying for a Dutch male individual currently aged 25 in 41 year from now. The lower right panel corresponds with the parameters of the distribution of the probability of dying for a Dutch male individual currently aged 25 in 72 year from now. From the figure and the table we observe that the approximation is accurate. The approximation is less accurate when μ is close to zero or very small, σ is high, and in the tails of the distribution.

Figure 3.A.2: **Comparison (1 − Log Normal) - Log Normal**



This figure displays the probability density function of $Y = 1 - X$, with $X \sim \log N(\mu_X, \sigma_X^2)$ and the probability density function of the approximation $Z \sim \log N(\mu_Z, \sigma_Z^2)$ for different values of $\mu \equiv \mu_X$ and $\sigma \equiv \sigma_X$. The parameters of Z are set such that the first two moments of Z match the first two moments of X.

Table 3.A.2: **Comparison (1 − Log Normal) - Log Normal**

	μ_X = -0.052786; σ_X = 0.0048811		μ_X = -0.31694; σ_X = 0.022088		μ_X = -0.38612; σ_X = 0.023094		μ_X = -3.5138; σ_X = 0.064043	
Q	X	Z	X	Z	X	Z	X	Z
0.01	0.041	0.042	0.233	0.236	0.283	0.285	0.965	0.966
0.025	0.042	0.043	0.239	0.241	0.289	0.290	0.966	0.966
0.05	0.044	0.044	0.245	0.246	0.294	0.295	0.967	0.967
0.1	0.045	0.046	0.251	0.251	0.300	0.300	0.968	0.968
0.25	0.048	0.048	0.261	0.260	0.310	0.309	0.969	0.969
0.5	0.051	0.051	0.272	0.271	0.320	0.320	0.970	0.970
0.75	0.055	0.054	0.282	0.282	0.331	0.330	0.971	0.971
0.9	0.057	0.057	0.292	0.292	0.340	0.340	0.973	0.973
0.95	0.059	0.059	0.298	0.299	0.346	0.347	0.973	0.973
0.975	0.060	0.061	0.302	0.304	0.350	0.352	0.974	0.974
0.99	0.062	0.063	0.308	0.311	0.356	0.358	0.974	0.975

This table displays the quantiles of the distribution $Y = 1 - X$, with $X \sim \log N(\mu_X, \sigma_x^2)$ and the quantiles of the approximating distribution $Z \sim \log N(\mu_Z, \sigma_Z^2)$. The parameters of Z are set such that the first two moments of Z match the first two moments of X. The first column with heading Q displays the quantiles. The other columns show the quantiles of X and Z for different values of μ_X and σ_X, where the subcolumns with heading X present the quantiles of the distribution of X, and the subcolumns with heading Z the quantiles of the corresponding distribution of Z.

Payments. We proceed with the vector of time-specific liability payments $\widetilde{L}$, introduced in subsection 3.4.2. In term of the notation of the appendix we have

$$\widetilde{L}\left(sl, x, g, p, y_p, g_p, s\right) = 1_{[r_j,\infty)}(T) \times \widetilde{S}\left(x, s, g\right)$$

for a single-life annuity (with $r_j = \max\{65 - x, 1\}$, the number of years until the first single life annuity payment), and

$$\widetilde{L}\left(sa, x, g, p, y_p, g_p, s\right) = p \times \widetilde{D}\left(x, s, g\right) \widetilde{S}\left(y_p s, g_p\right)$$

for a survivor annuity. Each component $\widetilde{L}\left(j\right)$, $j \in \mathcal{J}$, is of the form

$$\widetilde{L}\left(j\right) = c_j \left(\prod_{u \in U_j} H\left(u\right)\right), \tag{3.36}$$

where U_j is the relevant index set for the components of H to be included, and the constants c_j is $1_{[r_j,\infty)}(s)$ or p.

Given $H \sim \log N\left(\mu_{\ell H}, \Sigma_{\ell H}\right)$, we have

$$\begin{pmatrix} \widetilde{L} \\ H \end{pmatrix} \sim \log N\left(\begin{pmatrix} \mu_{\ell L} \\ \mu_{\ell H} \end{pmatrix}, \begin{pmatrix} \Sigma_{\ell L} & \Sigma_{\ell LH} \\ \Sigma_{\ell HL} & \Sigma_{\ell H} \end{pmatrix}\right),$$

with for $j \in \mathcal{J}$,

$$\mu_{\ell L}(j) = \log\left(c_j\right) + \sum_{u \in U_j} \mu_{\ell H}\left(u\right),$$

and for $j_1, j_2 \in \mathcal{J}$,

$$\Sigma_{\ell L}\left(j_1, j_2\right) = \sum_{u_1 \in U_{j_1}} \sum_{u_2 \in U_{j_2}} \Sigma_{\ell H}\left(u_1, u_2\right),$$

and

$$\Sigma_{\ell HL}\left(j_1, j_2\right) = \Sigma_{\ell LH}\left(j_2, j_1\right) = \sum_{u_2 \in U_{j_2}} \Sigma_{\ell H}\left(j_1, u_2\right).$$

Third approximation. The time $(t+s)$-specific discounted portfolio payments $\widetilde{L}_{t+s}$ are just a linear transformation of $\widetilde{L}$, i.e., we can write $\widetilde{L}_{t+s} = \Delta_s \widetilde{L}$, for some appropriate Δ_s, depending on the products included. More generally, we have

$$\begin{pmatrix} \widetilde{L}_{t+1} \\ \vdots \\ \widetilde{L}_{t+T} \end{pmatrix} = \begin{pmatrix} \Delta_1 \\ \vdots \\ \Delta_T \end{pmatrix} \widetilde{L} \equiv \Delta \widetilde{L}.$$

We write $\widetilde{L}_\Delta = \Delta \widetilde{L}$. As third approximation, we apply the Fenton-Wilkinson approximation with $A = \widetilde{L}_\Delta$ and $G = H$, so that this joint vector, by approximation and conditional upon $\mathcal{F}_t$, follows a lognormal distribution, with parameters following from the Fenton-Wilkinson approximation.

Accuracy Approximation Liabilities. In order to indicate the approximation error in the approximation steps so far, we present in Table 3.A.3 the 95% confidence intervals of the simulation uncertainty for three quantiles of the distribution of the discounted cash flows of appropriate components of $\widetilde{L}$, corresponding to four products, discounted with an interest rate of 4%.

Table 3.A.3: **95 percent confidence intervals of quantiles of the distributions**

	N=10,000	N=100,000	Model
	Q(0.75)		
sl m	[11.6048 - 11.6126]	[11.6066 - 11.6093]	11.6059
sl f	[13.2760 - 13.2872]	[13.2826 - 13.2865]	13.2824
sa m	[3.3288 - 3.3373]	[3.3293 - 3.3317]	3.3313
sa f	[1.6605 - 1.6658]	[1.6642 - 1.6658]	1.6666
	Q(0.90)		
sl m	[11.6959 - 11.7068]	[11.7007 - 11.7039]	11.7038
sl f	[13.4047 - 13.4171]	[13.4166 - 13.4212]	13.4256
sa m	[3.4154 - 3.4245]	[3.4182 - 3.4216]	3.4245
sa f	[1.7197 - 1.7257]	[1.7221 - 1.7241]	1.7254
	Q(0.995)		
sl m	[11.8873 - 11.9240]	[11.8918 - 11.9018]	11.9152
sl f	[13.6767 - 13.7078]	[13.6861 - 13.6982]	13.7361
sa m	[3.5967 - 3.6266]	[3.6084 - 3.6175]	3.6319
sa f	[1.8451 - 1.8636]	[1.8543 - 1.8593]	1.8577

This table displays the 95% confidence intervals of the simulation uncertainty for three quantiles of the distribution of the discounted cash flows of $\widetilde{L}$ based on 10,000 (given in the column “N=10,000”) and 100,000 (given in the column “N=100,000”) simulations and the approximated value of the quantiles of the different products using approximations 1–3 (given in the column “Model”). The three quantiles are the 75%, 90%, and the 99.5% quantile, the payments are discounted with an interest rate of 4%. The products are “sl m,” a single life annuity for a Dutch male aged 65, “sl f,” a single life annuity for a Dutch female aged 65, “sa m,” a survivor annuity for a Dutch male aged 65 with a Dutch female partner aged 65, and “sl m,” a survivor annuity for a Dutch female aged 65 with a Dutch male partner aged 65, and the yearly discount rate is 4%.

The four products are a single life annuity for a Dutch male aged 65, a single life annuity for a Dutch female aged 65, a survivor annuity for a Dutch male aged 65 with a Dutch female partner aged 65, and a survivor annuity for a Dutch female aged 65 with a Dutch male partner aged 65 The three quantiles are the 75%, 90%, and the 99.5% quantile, based on 10,000 and 100,000 simulations. We observe that the approximated value of the quantile of the different products are close to, or even within, the 95% confidence intervals of the simulation uncertainty. This even holds for the higher quantiles and for the products with more uncertainty (i.e., survival annuities). In contrast to the survival probabilities, we obtain a good approximation for all products, because the products are a sum of liability payments, where for most payments the uncertainty is not too large.

Cost of Capital. We start from $\widetilde{L}_{\Delta} = \Delta \widetilde{L}$, together with $L_{t+T+1} \equiv 0$. We shall now present approximations for L_{t+s}, $s \leq T$, as defined recursively by (3.11). These approximations will be denoted by $\widehat{L}_{t+s}$. We set $\widehat{L}_{t+T+1} = L_{t+T+1} = 0$. In (3.11) we make use of information sets $\mathcal{F}_{t+s}$. Given our approximation that $\widetilde{L}_{\Delta}$ jointly with H follows a lognormal distribution, this information set is the sigma-field generated by all the components of $\widetilde{L}_{\Delta}$ and H, realized before or at time $t+s$.

Fourth approximation. As our fourth approximation we take

$$\mathcal{F}_{t+s} = \sigma\left(H_s\right),$$

where H_s is the vector including as components $\widetilde{p}(x,s,g)$ and $\widetilde{S}(x,s,g)$, $x \in \mathcal{X}_0$ and $g \in \{M, F\}$ (where $\mathcal{X}_0 \subset \mathcal{X}$ represents the set of ages of the included insureds). This approximation is motivated by the Markov-property of $\widetilde{p}(x,s,g)$ and $\widetilde{S}(x,s,g)$, for $s \in \mathcal{T}_x$. The components of $\widetilde{L}_{\Delta}$ and H related to some time $t+s+\tau$ are transformations of $\widetilde{p}(x,s+\tau,g)$ and $\widetilde{S}(x,s+\tau,g)$, $x \in \mathcal{X}_0$, and $g \in \{M,F\}$. Using then $\sigma\left(H_s\right)$, with H_s as defined, then seems to yield a good approximation when the actual conditioning is on $\mathcal{F}_{t+s}$.

Given this fourth approximation, our approximations of $\widehat{L}_{t+s}$ are based on the following lemma, that follows from straightforward calculations.

Lemma 3.A.2 *Suppose*

$$\begin{pmatrix} A \\ G \end{pmatrix} \sim \log N\left(\begin{pmatrix} \mu_{\ell A} \\ \mu_{\ell G} \end{pmatrix}, \begin{pmatrix} \Sigma_{\ell A} & \Sigma_{\ell AG} \\ \Sigma_{\ell GA} & \Sigma_{\ell G} \end{pmatrix}\right),$$

where $\Sigma_{\ell G}$ is invertible, then, with a and g the componentwise log-s of A and G, respectively, we have

$$\begin{aligned}
&c\mathbb{E}\left(A \mid G\right) + d\mathbb{Q}_{1-\alpha}\left(A \mid G\right) = \\
&\quad c\exp\left(E\left(a \mid g\right) + \frac{1}{2}Var\left(a \mid g\right)\right) + d\exp\left(E\left(a \mid g\right) + \Phi^{-1}(1-\alpha)\sqrt{Var\left(a \mid g\right)}\right) \\
&= F\exp\left(E\left(a \mid g\right)\right),
\end{aligned}$$

where

$$E(a \mid g) = \mu_{\ell A} + \Sigma_{\ell AG}\Sigma_{\ell G}^{-1}(g - \mu_{\ell G}),$$
$$Var(a \mid g) = \Sigma_{\ell A} - \Sigma_{\ell AG}\Sigma_{\ell G}^{-1}\Sigma_{\ell GA},$$

and

$$F = c\exp\left(\frac{1}{2}Var(a \mid g)\right) + d\exp\left(\Phi^{-1}(1-\alpha)\sqrt{Var(a \mid g)}\right).$$

We apply lemma 3.A.2 to the last period, i.e., we take $A = \widetilde{L}_{t+T}$, $G = H_T$, $c = 1/(1 + CoC)$, and $d = (1 - c)/\left(1 + r^{rf}\right)$. Then, based on the approximations used so far, we obtain as approximation

$$\begin{aligned}\widehat{L}_{t+T} &= F\exp\left(\mu_{\ell A} + \Sigma_{\ell AG}\Sigma_{\ell G}^{-1}(g - \mu_{\ell G})\right)\\ &= \exp\left(f + \mu_{\ell A} + \Sigma_{\ell AG}\Sigma_{\ell G}^{-1}\log(H_T)\right),\end{aligned} \tag{3.37}$$

with $f = \log(F)$. Given our approximations, $\widehat{L}_{t+T-1}$ is also lognormal: starting from $H_T \mid \mathcal{F}_t \sim \log N(\mu_{\ell G}, \Sigma_{\ell G})$, we find

$$\widehat{L}_{t+T} \mid \mathcal{F}_t \sim \log N\left(f + \mu_{\ell A}, \Sigma_{\ell AG}\Sigma_{\ell G}^{-1}\Sigma_{\ell GA}\right),$$

Using lemma 3.A.2 once again, but now with $c = 0$, $d = 1/\left(1 + r^{rf}\right)$, $A = \widetilde{L}_{t+T}$, and $G = H_T$, we find the approximate distribution of the first term of the right hand side of (3.13) for $s = T$. Taking the difference between this approximation and (3.37), we find the approximation of SCR_{t+T}, and then also the approximation of $MVM_{t+T} \equiv CoC \cdot SCR_{t+T}$. In addition, we can then easily calculate the approximations of $E_{t+s}[SCR_{t+T}]$, needed to calculate MVM_{t+s}, for each $s \leq T$. Finally, BEL_{t+T} can be approximated using (3.12), for $t = T$.

Fifth approximation. Next, we present the iteration step to find the approximations $\widehat{L}_{t+s}$, for $s = T - 1$ down to $s = 0$, starting from the approximation $\widehat{L}_{t+T}$. We illustrate the iteration step for $s = T$ to $s = T - 1$. Given our approximations and conditional upon $\mathcal{F}_t$, the vector $\left(\widehat{L}_{t+T}, \widetilde{L}_{t+T-1}, H'_{T-1}\right)'$ is lognormal, with mean vector and covariance matrix that can easily be calculated using (3.37) and using that the vector $\left(\widetilde{L}_{t+T-1}, H'_T, H'_{T-1}\right)'$ is a subvector of the vector $\left(\widetilde{L}'_{\Delta}, H'\right)'$, which (given our approximations and conditional upon $\mathcal{F}_t$) is lognormal. As fifth approximation, we apply the Fenton-Wilkinson approximation to

$$\begin{pmatrix} \widetilde{L}_{t+T-1} + \widehat{L}_{t+T} \\ H_{T-1} \end{pmatrix},$$

so that by approximation and conditional upon $\mathcal{F}_t$ this vector follows a lognormal distribution. Given this approximation, we can use lemma 3.A.2, with now $A = \widetilde{L}_{t+T-1} + \widehat{L}_{t+T}$ and $G = H_{T-1}$, to obtain the approximation $\widehat{L}_{t+T-1}$, just like (3.37) is obtained in the first step. In addition, we can then calculate the approximation of SCR_{t+T-1}, and the approximation of MVM_{t+T-1}, using here also the approximation of $E_{t+T-1}[SCR_{t+T}]$ from the previous stage. Given the approximation of SCR_{t+T-1}, we can calculate the approximations of $E_{t+s}[SCR_{t+T-1}]$ needed to calculate MVM_{t+s}, for each $s \leq T-1$. The iteration procedure can then be repeated to obtain the approximations $\widehat{L}_{t+s}$, and approximations for SCR_{t+s} and MVM_{t+s}, $s \leq T-1$. Finally, BEL_{t+T-1} can be approximated using (3.12), for $t = T-1$.

Accuracy Cost of Capital approach. In order to quantify the effect of the approximations, ideally one should compare the probability distribution of the original L_{t+s}, $s \geq 1$, as induced by the Lee-Carter based probability distribution $\mathbb{P}_t$, with the approximate distribution of $\widehat{L}_{t+s}$, $s \geq 1$. However, it would require too many simulations from $\mathbb{P}_t$ to calculate the distribution of the original L_{t+s}, $s \geq 1$, in a sufficiently accurate way. Instead, Table 3.A.4 displays the difference in the discounted sum of the SCR-s multiplied with the cost of capital rate, i.e., the MVM, in the best estimate scenario, using the approximate distribution and using alternative simulation-based approximations of $\mathbb{P}_t$ that are obtained in the following way:

i) Given the median scenario of the survival probabilities until time $t+s$ (denoted by ω_{t+s}^{ME}), we simulate N times the distribution of the survival probabilities at time $t+s+1$ and the corresponding liability payments $\widetilde{L}_{t+s+1}$.

ii) For each of the simulated survival probabilities in step i), the market value of the liabilities at time $t+s+1$, $\widehat{L}_{t+s+1}|\omega_{t+s}^{ME}$, is approximated using the the version of (3.37) applicable to period $t+s+1$.

iii) The time-$(t+s)$ value $L_{t+s}|\omega_{t+s}^{ME}$ is obtained by means of (3.11), using $\widehat{L}_{t+s+1}|\omega_{t+s}^{ME}$ from ii) as approximation for L_{t+s+1}.

iv) Finally, $\widehat{L}_{t+s}|\omega_{t+s}^{ME}$ is obtained using the version of equation (3.37), applicable to time $t+s$.

Table 3.A.4 displays the aggregated discounted sum of the differences between $L_{t+s}|\omega_{t+s}^{ME}$ from step iii) and $\widehat{L}_{t+s}|\omega_{t+s}^{ME}$ from step iv), for $s = 0, \ldots, T$, for the four different portfolios as described in Section 3.5.

Table 3.A.4: **Table with approximation error**

Portfolio	1	2	3	4
	US			
CoC=0.06	1.09%	1.10%	1.07%	0.95%
CoC=0.04	1.02%	1.41%	0.57%	1.37%
	UK			
CoC=0.06	0.49%	0.77%	0.68%	0.87%
CoC=0.04	0.45%	0.76%	0.37%	1.32%
	NL			
CoC=0.06	1.62%	0.35%	0.73%	0.80%
CoC=0.04	1.80%	0.52%	0.90%	0.61%

This table displays the aggregated discounted difference between simulations and approximation relative to MVM_t. The values are given for the different portfolios as described in Section 3.5 for insureds in the US (US), UK (UK), and the Netherlands (NL), and using a Cost of Capital rate of 6% (CoC=0.06) and of 4% (CoC=0.04).

3.A.2 Estimation of parameters of the Lee and Carter (1992) model

We estimated the parameters of the Lee and Carter (1992) model in the following way. Age (x), gender (g), and time (τ) specific numbers of death ($D_{x,\tau}^{g}$) and exposed to death ($E_{x,\tau}^{g}$) are obtained from the Human Mortality Database.[13] The force of mortality, as given in equation (3.20), is given by

$$\mu_{x,\tau}^{g} = \frac{D_{x,\tau}^{g}}{E_{x,\tau}^{g}}. \tag{3.38}$$

Using the forces of mortality, we first estimate a_x^g by the mean of the log of the forces of mortality

$$a_x^g = \frac{\sum_{\tau=1970}^{2006} \log\left(\mu_{x,\tau}^{g}\right)}{30}. \tag{3.39}$$

[13] See www.mortality.org.

Then we apply a Singular Value Decomposition (SVD) to Z^g, the matrix with components $Z^g(x,\tau) = \log\left(\mu^g_{x,\tau}\right) - a^g_x$ the matrix of the logarithms of the rates after the averages over time of the log age-specific rates have been subtracted. The SVD of Z^g is given by

$$Z^g = U^g \Sigma^g V^g, \tag{3.40}$$

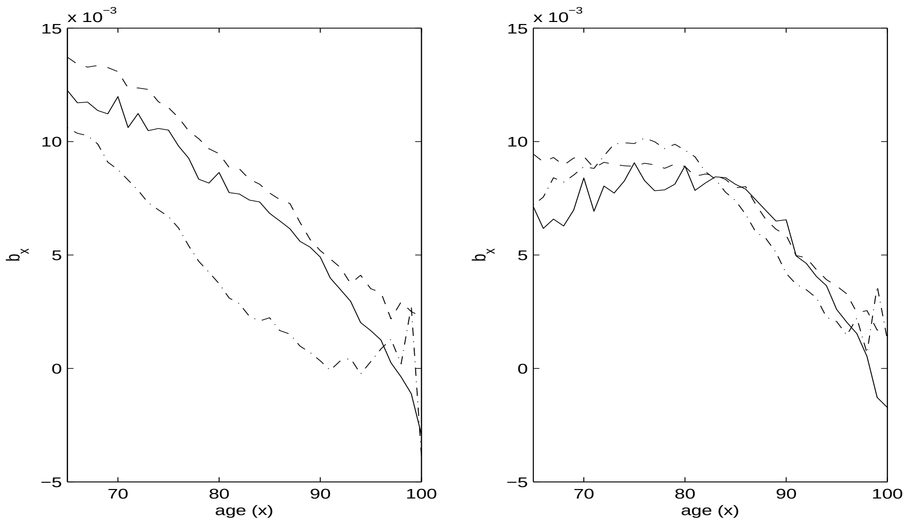
where U^g is an 111×111 unitary matrix, Σ^g is an 111×37, and V^g is an 37×37 unitary matrix. Let σ^g_1 be the leading singular value of Σ^g, u^g_1 the corresponding column vector of U^g, and v^g_1 the corresponding row vector of V^g, then we estimate the parameters b^g_x and k^g_τ by

$$b^g_x = \frac{u^g_1(x+1)}{\sum_{i=1}^{111} u^g_1(i)} \tag{3.41}$$

$$k^g_\tau = \sigma^g_1 \cdot v^g_1(\tau) \cdot \sum_{i=1}^{111} u^g_1(i), \tag{3.42}$$

hence, b^g_x and k^g_τ are are obtained by he first right and left vectors and leading value of the SVD, after the normalization $\sum b^g_x = 1$ and $\sum k^g_\tau = 0$, providing a unique solution. Figure 3.A.3 displays the parameter estimate of b_x in the Lee-Carter model for the US, UK, and the Netherlands. The parameters of the random walk with drift, as given in equation (3.22), are estimated using OLS. Table 3.A.5 displays the parameter estimates of random walk with drift in the Lee-Carter model for the US, UK, and the Netherlands.

Figure 3.A.3: **Parameter b_x in the Lee-Carter model**



The figure displays the estimate of b_x in the Lee-Carter model for the US (solid curve), UK (dashed curve), and the Netherlands (dashed-dotted curve). The left panel is for males, the right panel is for females. The parameter is estimated using the mortality data from 1970 till 2006.

Table 3.A.5: **Parameter estimates of the Lee-Carter model**

parameter	US M	US F	UK M	UK F	NL M	NL F
c^g	-1.5875	-1.3652	-1.6429	-1.5975	-1.8741	-1.5525
σ_c^g	0.2097	0.2320	0.2357	0.3568	0.3753	0.5344
σ_e^g	1.2583	1.3920	1.4141	2.1406	2.2520	3.2063
ρ_c	0.7991		0.7661		0.3448	

The table displays the estimates of the Lee-Carter model in the US, UK, and the Netherlands. The parameters are estimated using the mortality data from 1970 till 2006.

Chapter 4

The effect of product design

This chapter is based on Stevens, De Waegenaere and Melenberg (2010a).

4.1 Introduction

Life expectancy of both males and females has increased substantially over the past decades. The increase in life expectancy imposes a burden on pension providers, since it induces an increase in the duration of pension payments. For example, over the past three decades, the remaining life expectancy of a male Dutch retiree aged 65 has increased by on average one year per decade. More importantly, there is considerable uncertainty regarding the future development of life expectancy. Existing literature shows that this uncertainty, which is referred to as *longevity risk*, potentially imposes significant risk on pension funds and insurers. For example, Hári, De Waegenaere, Melenberg, and Nijman (2008b) show that the buffer that is required to reduce the probability of underfunding to an acceptable level increases significantly when longevity risk is taken into account. This uncertainty is a major concern to pension funds and regulators. At present, there is considerable interest in the development of longevity-linked financial instruments such as, for example, longevity bonds. Since the payoff of such securities is linked to the development of survival rates, they could be used to partially hedge longevity risk in pension and life insurance portfolios. There is currently a large body of literature that focusses on the design and valuation of longevity-linked securities (see, for example, Blake et al. 2006, and references therein). However, until now, attempts to generate trade have not been successful. The focus in this chapter is on an alternative tool to manage longevity risk in pension annuities, namely through product design.

Existing literature mainly focuses on the effect of longevity risk on single life annuities (see, e.g., Olivieri 2001, Olivieri and Pitacco 2003, Cossette et al. 2007, and Hári et al. 2008b). However, many defined benefit pension funds offer both old-age pension insurance and partner pension insurance.[1] The former consists of a *single life annuity* for the life of the participant. The latter consists of a *survivor annuity* for the life of the partner, if the partner outlives the participant. The combination of a single life annuity and a survivor annuity is referred to as a *joint and survivor annuity*. Pension funds typically allow participants to choose, at retirement date, between a single life annuity and a joint and survivor annuity. A crucial design aspect of such plans is how pension rights are accrued. Two alternatives exist. In a *JointLife* plan, the participant builds up the right to receive a *joint and survivor annuity*. At retirement date, the participant has the option to exchange this annuity for a single life annuity with a higher annual payment. In a *SingleLife* plan, the participant builds up the right to receive a single life annuity. At retirement date, the participant has the option to exchange part of this annuity for a survivor annuity. In both types of plans, the *conversion rate*, i.e., the rate at which the participant will be able to exchange one type of annuity for the other type, has to be actuarially neutral at the time of exchange. Actuarial neutrality requires that the expected present value of the liabilities before exchange equals the expected present value of the liabilities after exchange.

Our goal in this chapter is to investigate the effect of longevity risk on the liabilities of these two types of pension plans, and how this effect depends on the design of the pension plan. In both types of pension plans, the sensitivity of the liabilities to longevity risk is driven by two effects. First, survivor annuities to some extent provide a natural hedge for single life annuities. Therefore, the sensitivity of the plan's liabilities to longevity risk is likely to depend strongly on product mix, i.e., the ratio of survivor benefits over single life benefits. Because the participants' choices at retirement date affect product mix, the exchange option affects the natural hedge potential that arises from combining single life and survivor annuities. Second, for insureds who did not yet reach retirement age, the conversion rate is uncertain because it depends on survival rates at the time of exchange. As a consequence, the exchange option induces uncertainty in the *level* of the payments, as well as

[1] The Retirement Equity Act of 1984 (REA) amended the Employee Retirement Income Security Act of 1974 (ERISA) to introduce mandatory spousal rights in pension plans.

dependence between the level of payments and the *duration* of payments. We find that whereas product mix effects are identical in both plans, dependence effects are fundamentally different. In contrast to the *SingleLife* plan, in a *JointLife* plan a higher than expected duration of single life annuity payments is partly mitigated by a lower than expected nominal level of the payment. Therefore, the *JointLife* plan is significantly less sensitive to longevity risk than the *SingleLife* plan.

This chapter is organized as follows. In Section 4.2, we formally define the two types of pension plans, and introduce some basic concepts and notation. In Section 4.3, we determine the actuarially neutral conversion rate for the exchange of survivor annuity rights for additional single life annuity rights, or vice versa. In Section 4.4, we show how conversion rate risk affects the nominal insured rights for the single life annuity and the survivor annuity, respectively, for both types of pension plans. Section 4.5 combines the results of the previous two sections to determine the liabilities in each type of pension plan. Section 4.6 investigates how the sensitivity to longevity risk depends on the design of the plan. Section 4.7 concludes.

4.2 The model

Our goal in this chapter is to investigate the effect of longevity risk on the liabilities of pension plans that offer both old-age pension protection and partner pension protection. In particular, we investigate whether the adverse effects of longevity risk on portfolios of single life and survivor annuities can be mitigated through product design. In Subsection 4.2.1, we formally define the pension plans. In Subsection 4.2.2, we introduce the model and techniques that will be used to quantify longevity risk.

4.2.1 Pension plan definition

We consider two types of pension plans that, at retirement date, allow the participant to choose between a single life annuity and a joint and survivor annuity. The two plans differ with respect to the accrual of annuity rights. Formally, the two plans are defined as follows:

- In a *JointLife* plan, the participant accrues the right to receive a joint and survivor annuity, consisting of a deferred single life annuity with a yearly nominal payment of 1, combined with a deferred survivor annuity with a yearly nominal payment of w. At retirement date, the participant has the option to exchange, at an actuarially neutral rate, the survivor annuity for additional single life annuity.

- In a *SingleLife* plan, the participant accrues the right to receive a *single life annuity* with a yearly nominal payment of 1. At retirement date, the participant has the option to exchange, at an actuarially neutral rate, part of the single life annuity for a survivor annuity. After exchange, the ratio of the nominal payment of the survivor annuity over the nominal payment of the single life annuity should be equal to w.

For both types of pension plans, the liabilities consist of a stream of payments in future periods. Throughout this chapter, we consider a fixed and given year t, and quantify the uncertainty in the payments in future years $t+\tau$, for a participant with given characteristics. Because in any future period, the level of the payment depends on whether the participant and/or the partner is alive, relevant characteristics are those that affect the probability distribution of their remaining lifetimes. In addition to age, remaining lifetime also significantly depends on gender and time. Indeed, women have higher life expectancy than men, and past trends suggest that remaining life expectancy of future retirees might be substantially higher than the remaining life expectancy of current retirees. Therefore, we introduce the following notation:

- A participant is characterized by a vector $((x,g),p,(y,g'))$, where x denotes his (her) age, $g \in \{m,f\}$ denotes the gender, $p=1$ ($p=0$) if the insured does (does not) have a partner, and, if $p=1$, y denotes the age of the partner, and $g' \in \{m,f\}$ denotes the gender of the partner;

- $T^{(g)}_{x,t}$ denotes the *remaining lifetime* at time t of an individual with gender $g \in \{m,f\}$ and age x at time t. Furthermore, we denote $1_{(T^{(g)}_{x,t} \geqslant \tau)}$ for the indicator random variable that equals 1 if an individual (participant or partner) with gender g and age x at time t will survive at least τ more years, and zero otherwise.

Then, the payments for a single life and a survivor annuity, respectively, are defined as follows (see, also, e.g., Gerber 1997):

i) A *(deferred) single life annuity*, which yields a yearly payment of 1 in every year that the participant is alive and older than 65, i.e., the payment in period $t+\tau$ equals

$$\widetilde{L}_{sl}(x,g,\tau,t) = 1_{(T^{(g)}_{x,t} \geqslant \tau)} 1_{(x+\tau \geqslant 65)}, \text{ for } \tau = 0,1,\dots \tag{4.1}$$

ii) A *(deferred) survivor annuity*, which yields a yearly payment of 1 in every year that the spouse outlives the participant, in case the participant dies after retirement age. The payment in period $t+\tau$ equals:

$$\widetilde{L}_{surv}(x,y,g,g',\tau,t) = 1_{(65-x \leqslant T^{(g)}_{x,t} < \tau)} 1_{(T^{(g')}_{y,t} \geqslant \tau)}, \text{ for } \tau = 0,1,\dots \tag{4.2}$$

In each plan, the random payment at a future date to a given participant will be either a single life annuity payment or a survivor annuity payment, but the nominal rights for each type of annuity depend on:

i) the conversion rate, i.e., the rate at which the participant will be able to exchange one type of annuity for the other type;

ii) the type of pension plan;

iii) the participant's choice at retirement date between a single life annuity and a joint and survivor annuity.

In section 4.3, we determine the actuarially neutral conversion rate for the exchange of survivor annuity rights for additional single life annuity rights, or vice versa. In Section 4.4, we determine the nominal rights as a function of the participant's choice, for both types of pension plans.

4.2.2 Quantifying longevity risk

As argued above, our focus in this chapter is on how the liabilities of pension plans that offer both old-age insurance and partner pension insurance are affected by uncertainty in the remaining lifetimes of the participant and/or the partner, and on how that effect depends on the design of the pension plan. We consider a participant characterized by $((x,g),p,(y,g'))$ at date t. To avoid overloaded notation, we do not explicitly denote age-, gender-, and time- dependence of the liabilities, and denote:

- $\widetilde{L}_{sl}(\tau)$ and $\widetilde{L}_{surv}(\tau)$, for the normalized single life and survivor annuity payment at date $t+\tau$, as defined in (4.1) and (4.2), respectively;
- $\widetilde{L}(\tau)$ for the payment to the participant at date $t+\tau$, in a *JointLife* plan or a *SingleLife* plan.

There are several alternative approaches to measure risk in liabilities that consist of a stream of payments at future dates (see, e.g., Olivieri and Pitacco 2003). Our measure of longevity risk is the extent to which the *present value* of future payments is affected by uncertainty in the remaining lifetimes of the participant and his partner, i.e., we focus on the probability distribution of:[2]

$$\widetilde{L} := \sum_{\tau=0}^{110-x} \widetilde{L}(\tau) \cdot \frac{1}{(1+r)^{\tau}}, \tag{4.3}$$

where r denotes the risk free interest rate. The $(1-\epsilon)\%$ quantile of the distribution of $\widetilde{L}$ is equal to the amount of money that, when invested at the risk free rate, is needed to guarantee that the probability that in some future period there will be insufficient funds to cover the liabilities is less than $\epsilon\%$.

The present value of the future payments is affected by two types of longevity risk:

- *non-systematic longevity risk,* because $\widetilde{L}(\tau)$ depends on the remaining lifetime of the participant and/or his partner, and, conditional on given survival probabilities, these remaining lifetimes are random variables;
- *systematic longevity risk,* because survival probabilities for future dates are random variables.

However, it is well-known that non–systematic longevity risk becomes negligible in large pools (see, e.g., Olivieri 2001; Olivieri and Pitacco 2003; and H'ari et al. 2008b). In contrast, systematic longevity risk does not decrease with portfolio size. Our focus therefore is on the effect of systematic longevity risk on the two types of pension plans. Then, the random variable of interest is given by:

$$L := \sum_{\tau=0}^{110-x} \mathbb{E}\left[\widetilde{L}(\tau)|\mathcal{F}_{\infty}\right] \frac{1}{(1+r)^{\tau}}, \tag{4.4}$$

[2]We assume that the probability that an insured reaches the age of 111 is negligibly small. Our focus is on the relative importance of longevity risk in the two types of plans. We ignore interest rate risk.

where $\mathcal{F}_\infty$ denotes the set of all future death rates. Thus, L is a random variable that depends on future death rates, and our goal is to investigate how the probability distribution of L depends on the type of plan. In order to do so, we proceed as follows:

1. We determine L as a linear combination of the present value of payments of a normalized single life annuity and a survivor annuity, which are given by:

$$\begin{aligned} L_{sl} &= \sum_{\tau=0}^{110-x} \mathbb{E}\left[\widetilde{L}_{sl}(\tau)|\mathcal{F}_\infty\right] \frac{1}{(1+r)^\tau} \\ &= \sum_{\tau=\max\{65-x,0\}}^{110-x} {}_\tau p_{x,t}^{(g)} \cdot \frac{1}{(1+r)^\tau}, \end{aligned} \tag{4.5}$$

   and[3]

$$\begin{aligned} L_{surv} &= \sum_{\tau=0}^{110-x} \mathbb{E}\left[\widetilde{L}_{surv}(\tau)|\mathcal{F}_\infty\right] \frac{1}{(1+r)^\tau}, \\ &= \sum_{\tau=\max\{65-x,0\}}^{110-y} \left({}_{\max\{65-x,0\}}p_{x,t}^{(g)} - {}_\tau p_{x,t}^{(g)}\right) \cdot {}_\tau p_{y,t}^{(g')} \cdot \frac{1}{(1+r)^\tau}, \end{aligned} \tag{4.6}$$

   respectively, where

   - $p_{x+s,t+s}^{(g)}$ for $s >= 0$ denote the realized one-year survival probabilities of the cohort aged x in year t, as defined in (2.5);
   - ${}_\tau p_{x,t}^{(g)} = p_{x,t}^{(g)} \cdot p_{x+1,t+1}^{(g)} \cdots p_{x+\tau-1,t+\tau-1}^{(g)}$ denotes the realized τ-years survival probability of the cohort aged x in year t.

   To determine L as a combination of L_{sl} and L_{surv}, we determine the rate at which one type of annuity can be exchanged for the other (Section 4.3), the nominal insured rights for the single life and the survivor annuity as a function of the participant's choice (Section 4.4), and the participant's choice at retirement date (Section 4.5).

[3]Existing literature shows that there exists dependence between the remaining lifetimes of a participant and his (her) partner at micro-level, e.g., due to the fact that partners have similar lifestyles, or that the passing away of a partner affects the surviving relative's quality of life. Because our focus in this chapter is on systematic longevity risk, we ignore this dependence and assume that the remaining lifetimes of the spouses, conditional on the survival probabilities, are independent.

2. We use stochastic forecast models to forecast the probability distribution of future survival probabilities $p_{x,s}^{(g)}$, for $s \geq t$. We include process risk, parameter risk, and model risk. To incorporate model risk, we estimate the models developed by Lee and Carter (1992), Brouhns, Denuit, and Vermunt (2002a), and Cossette et al. (2007). To estimate the parameters in each model, we use age-, gender-, and time-specific numbers of death and exposures to death for the Netherlands, obtained from the Human Mortality Database.[4] For a detailed description of the models, the estimation techniques, and the parameter estimates, we refer to Appendix 5.B.1.

4.3 The conversion rate

The *JointLife* plan includes the option to exchange survivor annuity rights for single life annuity rights. In contrast, the *SingleLife* plan includes the option to exchange single life annuity rights for survivor annuity rights. The liabilities of these pension plans therefore depend on the rate at which a single life annuity can be exchanged for a survivor annuity, and vice versa.

Pension laws typically prescribe that the exchange of one type of liability for another type of liability should be actuarially neutral. This implies that the conversion rate should be such that the expected present value of the liability payments after exchange equals the expected present value of the liability payments before exchange. While gender "discrimination" is typically not allowed by law (i.e., the conversion rate cannot depend on the gender of the insured and/or his (her) partner), pension laws vary in terms of how gender neutral liability values should be determined.[5] Some countries allow the conversion rate to depend on the age of the partner (such as, for example, the United States); others prohibit such age "discrimination" (for example, the Netherlands).

In this chapter we consider a gender- and age-neutral conversion rate. Because an insured aged x at date t has the option to exchange pension rights at retirement date, i.e., at date $\bar{t} = t + 65 - x$, the conversion rate for that insured will depend on gender- and age-neutral liability values at date $\bar{t}$. We let the gender- and age-neutral single life annuity, respectively survivor annuity, for a 65-year old

[4] www.mortality.org

[5] Defined benefit plans in the US are tax favored only if gender neutral values are determined on the basis of unisex life tables (see IRC section 417(e)(3)).

insured at date $\bar{t}$, be defined as the average of the liability for an insured with a three years younger partner, and an insured with a three years older partner, both with respect to gender-neutral death probabilities. Specifically, gender- and age-neutral liabilities are defined as:[6]

$$\begin{aligned} L_{sl,n,\bar{t}} :=& \quad L_{sl}(65, n, \bar{t}), \\ L_{surv,n,\bar{t}} :=& \quad \frac{1}{2} L_{surv}(65, 62, n, n, \bar{t}) + \frac{1}{2} L_{surv}(65, 68, n, n, \bar{t}), \end{aligned}$$

where $L_{sl}(65, n, \bar{t})$, $L_{surv}(65, 62, n, n, \bar{t})$, and $L_{surv}(65, 68, n, n, \bar{t})$ are given by (4.5) and (4.6), with $t = \bar{t}$, $x = 65$, $y = 62$ and $y = 68$, respectively, and with gender-specific death probabilities (${}_{\tau}p^{(g)}_{x,\bar{t}}$ and ${}_{\tau}p^{(g')}_{y,\bar{t}}$) replaced by gender-neutral probabilities ${}_{\tau}p^{(n)}_{x,\bar{t}}$, defined as:

$$ {}_{\tau}p^{(n)}_{x,\bar{t}} = \frac{1}{2}\left({}_{\tau}p^{(m)}_{x,\bar{t}} + {}_{\tau}p^{(f)}_{x,\bar{t}}\right), \text{ for all } x, \bar{t}, \tau. \tag{4.7}$$

Let us now consider the two types of plans.

i) In a *JointLife plan*, the participant can exchange survivor annuity rights for additional single life annuity rights. Actuarially neutral exchange requires that the expected value, at date $\bar{t}$, of the (gender- and age-neutral) liability after exchange is equal to the expected value, at date $\bar{t}$, of the (gender- and age-neutral) liability before exchange. Therefore, exchange of a nominal yearly survivor annuity right of 1 yields an (extra) nominal yearly single life annuity right of $e(\bar{t})$, where

$$ e\left(\bar{t}\right) := \frac{\mathbb{E}\left[L_{surv,n,\bar{t}} | \mathcal{F}_{\bar{t}}\right]}{\mathbb{E}[L_{sl,n,\bar{t}} | \mathcal{F}_{\bar{t}}]}, \tag{4.8}$$

and where

$$ \mathcal{F}_{\bar{t}} := \{p^{(g)}_{x,s} \mid g \in \{m, f\}, x \geqslant 0, \text{ and } s \leqslant \bar{t}\},$$

denotes the set of one-year death probabilities for all ages, both genders, and all time periods until time $\bar{t}$.

ii) In a *SingleLife plan*, the participant can exchange some single life annuity rights for additional survivor annuity rights. Actuarially neutral exchange, at date $\bar{t}$,

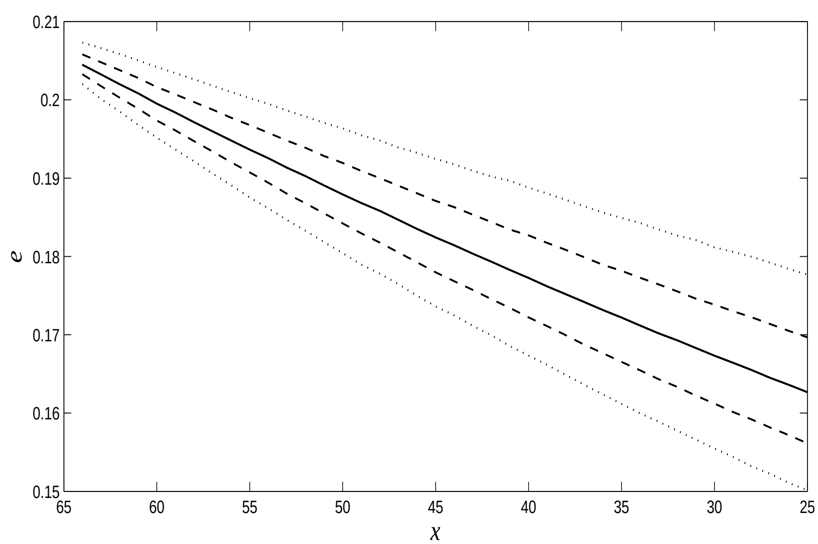
[6] The age difference is based on the average age difference in married couples (see, e.g. Brown and Poterba 2000). We have replicated our results with various alternative gender- and/or age-neutral conversion rates. In all cases, the qualitative results that we find in this chapter remain valid.

of a nominal yearly single life annuity right of 1 yields nominal yearly survivor annuity rights of $\frac{1}{e(\bar{t})}$.

Note that an insured aged $x < 65$ at date t will have the option to exchange annuity rights at retirement date, i.e., at date $\bar{t} = t + 65 - x > t$. Therefore, for all insureds who are younger than 65 at date t, the conversion rate $e(t + 65 - x)$ is a random variable. It is affected by longevity risk because it depends on death probabilities at time $\bar{t} = t + 65 - x > t$ (i.e., on the set $\mathcal{F}_{\bar{t}} = \mathcal{F}_{t+65-x}$).

To gain some insight in the extent of conversion rate uncertainty, Figure 4.3.1 displays selected quantiles of $e = e(t + 65 - x)$ as a function of x, where t is the present year.[7]

Figure 4.3.1: **Selected quantiles of the distribution of** $e(t + 65 - x)$



This figure displays selected quantiles of the distribution of $e(t + 65 - x)$, as a function of x: the median (bold line), the 25% and 75% quantile (dashed lines), and the 10% and 90% quantile (dotted lines).

We observe the following:

[7] The conversion rate can also be affected by interest rate risk. Using a one-factor Vasicek model, we found that uncertainty in the term structure of interest rates has a negligible effect on the distribution of $e(\bar{t})$. With a median trend in longevity, interest rate risk yields $\sigma(e(\bar{t}))/\mathbb{E}[e(\bar{t})] < 0.4\%$ for all ages between 25 and 64.

i) *The probability distribution of the conversion rate shifts downwards for younger cohorts.* This occurs because an increase in life expectancy of the participant implies an increase in the expected value of the single life annuity, $\mathbb{E}\left[L_{sl,n,\bar{t}}|\mathcal{F}_{\bar{t}}\right]$, but a decrease in the expected value of the survivor annuity, $\mathbb{E}\left[L_{surv,n,\bar{t}}|\mathcal{F}_{\bar{t}}\right]$. The latter occurs because the increase in life expectancy of the participant delays survivor annuity payments, so that they are more heavily discounted. Although increased life expectancy of the partner can lead to a longer duration of survivor annuity payments, the former effect is dominant.

ii) *For young insureds, the uncertainty in the conversion rate is substantial.* This occurs because exchange takes place at date $\bar{t} = t + 65 - x$, and the conversion rate depends on death probabilities at that time. For younger insureds, the exchange date $\bar{t}$ is further into the future, and therefore there is still considerable uncertainty regarding the death probabilities at time $\bar{t}$.

Conversion rate uncertainty has important consequences for both the participant and the pension provider. Depending on the type of plan and on the participant's choice at retirement date, the level of the nominal payments for the single life annuity and/or for the survivor annuity will depend on the conversion rate, which will only be realized at retirement date. Therefore, even though the longevity risk due to uncertainty in the duration of the payments is fully borne by the pension fund, the exchange option induces longevity risk for the participant too. Uncertainty in the conversion rate also affects the pension provider. First, it induces uncertainty in the product mix after retirement, i.e., the ratio of survivor benefits over single life benefits. Second, it induces dependence between the level of the payments and the duration of the payments. In the following section, we first investigate the effect of conversion rate uncertainty on the nominal insured rights.

4.4 Longevity risk in nominal rights

The previous section shows that there is considerable uncertainty regarding the conversion rate that will apply to insureds who are not yet retired. In this section we show how this uncertainty affects the nominal insured rights for the single life annuity and the survivor annuity at retirement date. We consider an insured characterized by $((x, g), p, (y, g'))$ at time t. In case the insured is still alive at retirement date, i.e., at date $\bar{t} = t + 65 - x$, he will need to choose between a single life annuity and a joint and survivor annuity. Let

$$1_{surv} \in \{0, 1\}, \tag{4.9}$$

be an indicator variable that, conditional on the insured being alive at time $\bar{t} = t + 65 - x$, equals 1 if he will choose a joint and survivor annuity, and 0 otherwise. In this section, we determine the nominal insured rights for a given choice at retirement date, i.e., we take 1_{surv} as given. In Section 4.5, we introduce a model for the participant's choice.

First, consider an insured in a *JointLife* plan. If the insured at time $\bar{t} = t+65-x$ will prefer a joint and survivor annuity ($1_{surv} = 1$), the nominal payment for the single life annuity equals $A_{sl} = 1$, and the nominal payment for the survivor annuity equals $A_{surv} = w$. If the insured will prefer to hold a single life annuity ($1_{surv} = 0$) (e.g., because he no longer has a partner), he exchanges survivor annuity rights for single life annuity rights. Since survivor annuity rights consist of a nominal yearly payment of w, actuarially neutral exchange of these rights yields additional single life annuity rights of $w \cdot e(\bar{t})$, where the conversion rate $e(\bar{t})$ is as defined in (4.8). Therefore, the nominal yearly payment of the single life annuity equals $A_{sl} = 1 + w \cdot e(\bar{t})$, and there is no survivor annuity, so $A_{surv} = 0$. Thus,

$$\begin{aligned} (A_{sl}, A_{surv}) \quad &= (1, w), & \text{if } 1_{surv} = 1, \\ &= (1 + w \cdot e(\bar{t}), 0), & \text{if } 1_{surv} = 0. \end{aligned} \tag{4.10}$$

Therefore, in case of a *JointLife* plan, uncertainty in the conversion rate $e(\bar{t})$ affects the level of the nominal payments in case the insured will prefer a single life annuity ($1_{surv} = 0$). Compared to a current retiree, a younger insured in a *JointLife* plan who prefers a single life annuity will, with high probability, receive less additional single life annuity rights ($w \cdot e(\bar{t})$) in exchange for survivor annuity rights (w). For example, when the ratio of insured rights for the survivor annuity over insured rights for the single life annuity equals $w = 2/3$, a current retiree would receive an increase of 13.64% in single life annuity rights in return for survivor annuity rights. The 95% confidence interval for a 25 year old is $[9.46\%, 12.79\%]$.

Next, consider an insured in a *SingleLife* plan. If the insured chooses a single life annuity ($1_{surv} = 0$), the nominal yearly payment for the single life annuity equals $A_{sl} = 1$, and there is no survivor annuity, so $A_{surv} = 0$. If the insured prefers a joint and survivor annuity ($1_{surv} = 1$), he exchanges some single life annuity rights for survivor annuity rights. The plan specifies that the ratio of survivor annuity rights over single life annuity rights after exchange has to be equal to w. This implies that, at date $\bar{t}$, a fraction $c(\bar{t})$ of single life annuity rights can be exchanged for $w \cdot \left(1 - c(\bar{t})\right)$ of survivor annuity rights. Because actuarially neutral exchange of a nominal yearly single life annuity right of $c(\bar{t})$ yields nominal yearly survivor annuity rights of $\frac{1}{e(\bar{t})} \cdot c(\bar{t})$, actuarial neutrality implies that:

$$\frac{1}{e\left(\bar{t}\right)} \cdot c(\bar{t}) = w \cdot \left(1 - c(\bar{t})\right),$$

or, equivalently,

$$c(\bar{t}) = \frac{w \cdot e\left(\bar{t}\right)}{1 + w \cdot e\left(\bar{t}\right)}. \tag{4.11}$$

Therefore, for an insured in a *SingleLife* plan who chooses a joint and survivor annuity, the nominal yearly payment for the single life annuity equals $A_{sl} = 1 - c(\bar{t}) = \frac{1}{1+w \cdot e(\bar{t})}$, and the nominal yearly payment for the survivor annuity equals $A_{surv} = w \cdot (1 - c(\bar{t})) = \frac{w}{1+w \cdot e(\bar{t})}$. Therefore,

$$\begin{aligned} (A_{sl}, A_{surv}) \quad &= \left(\tfrac{1}{1+w \cdot e(\bar{t})}, \tfrac{w}{1+w \cdot e(\bar{t})}\right), \qquad \text{if } 1_{surv} = 1, \\ &= (1, 0), \qquad \text{if } 1_{surv} = 0. \end{aligned} \tag{4.12}$$

In case of a *SingleLife* plan, uncertainty in the conversion rate induces uncertainty in both the level of the single life annuity as well as the survivor annuity in case the insured will prefer to hold a joint and survivor annuity ($1_{surv} = 1$). Compared to a current retiree, a younger insured in a *SingleLife* plan who will prefer a joint and survivor annuity will, with high probability, have to give up less single life annuity rights $\left(1 - \frac{1}{1+w \cdot e(\bar{t})} = \frac{w}{1+w \cdot e(\bar{t})} \cdot e(\bar{t})\right)$ and receive more survivor annuity rights $(\frac{w}{1+w \cdot e(\bar{t})})$. There is substantial uncertainty regarding the level of both the single life annuity and the survivor annuity rights. For example, when $w = 2/3$, a current retiree would have to give up 12.00% of single life annuity rights to receive survivor annuity rights. The 95% confidence interval for a 25 year old is $[8.64\%, 11.34\%]$.

4.5 Plan liabilities

In this section we use the results of the previous two sections to model the liabilities of a *JointLife* plan and of a *SingleLife* plan, respectively. Specifically, we express the present value of payments (conditional on death probabilities) in each pension plan as a linear combination of the present value of payments of a normalized single life annuity and survivor annuity, respectively.

We first determine the random payments in each plan conditional on a given choice 1_{surv} at retirement date, for participants with and without a partner.

In case of a *JointLife* plan, (4.10) implies that the payment at date $t+\tau$, is given by:

$$\begin{aligned} \widetilde{L}(\tau) &= 0, && \text{for } \tau < 65-x; \\ &= (1 + w \cdot e(\overline{t}) \cdot (1 - 1_{surv})) \cdot \widetilde{L}_{sl}(\tau), && \text{for } p = 0, \tau \geqslant 65-x; \\ &= \big(1 + w \cdot e(\overline{t}) \cdot (1 - 1_{surv})\big) \cdot \widetilde{L}_{sl}(\tau) + w 1_{surv} \widetilde{L}_{surv}(\tau), && \text{for } p = 1, \tau \geqslant 65-x, \end{aligned} \tag{4.13}$$

where $\widetilde{L}_{sl}(\tau)$ and $\widetilde{L}_{surv}(\tau)$ denote the random payment of a normalized single life and survivor annuity, respectively, as defined in (4.1) and (4.2).

In case of a *SingleLife* plan, (4.12) implies that:

$$\begin{aligned} \widetilde{L}(\tau) &= 0, && \text{for } \tau < 65-x; \\ &= \left(1 - 1_{surv} \cdot \tfrac{w \cdot e(\overline{t})}{1 + w \cdot e(\overline{t})}\right) \cdot \widetilde{L}_{sl}(\tau), && \text{for } p = 0, \tau \geqslant 65-x; \\ &= \left(1 - 1_{surv} \cdot \tfrac{w \cdot e(\overline{t})}{1 + w \cdot e(\overline{t})}\right) \cdot \widetilde{L}_{sl}(\tau) + 1_{surv} \tfrac{w}{1 + w \cdot e(\overline{t})} \widetilde{L}_{surv}(\tau), && \text{for } p = 1, \tau \geqslant 65-x. \end{aligned} \tag{4.14}$$

Now, in order to determine the present value of these payments conditional on death probabilities, it remains to specify the choice that the participant will (did) make at date $\overline{t} = t + 65 - x$, i.e., we need to specify the probability distribution of 1_{surv}. Clearly, that choice will also depend on whether the participant has a partner at the time the choice is made. In order to avoid unnecessary complications, we assume that a participant only loses a partner due to decease, and that participants without a partner will remain without a partner.[8] Then, we distinguish three cases:

i) For an insured aged $x < 65$ who does have a partner (i.e., $p = 1$), the choice will depend on whether the partner is alive at date $\overline{t} = t + 65 - x$. If the partner is not alive, the survivor annuity has no value, and the insured will

[8] Alternatively, one could replace $1_{(T_{y,t}^{(g')} \geqslant 65-x)} 1_{(p=1)}$ by $1_{(P_{x,t}^{(g)}=1)}$, where $P_{x,t}^{(g)}$ is an indicator random variable that indicates whether the participants will have a partner (not necessarily the current one) at retirement date $\overline{t} = t + 65 - x$.

prefer a single life annuity, and so $1_{surv} = 0$. If the partner is alive at time $\bar{t} = t + 65 - x$, the choice between a single life annuity and a joint and survivor annuity will depend on the couple's preference ordering. We let

$$1_F := 1_F((x,g),(y,g'),\bar{t}) \in \{0,1\}, \tag{4.15}$$

be an indicator variable that equals 1 if an insured with gender g aged 65 at time $\bar{t}$ with a partner with gender g' aged $65 - (x - y)$, prefers a joint and survivor annuity, and zero otherwise. To avoid overloaded notation, but to emphasize that the choice may depend on time because death probabilities change over time, we denote $1_F(\bar{t})$ for the preference indicator. Then,

$$1_{surv} = 1_F(\bar{t}) \cdot 1_{(T_{y,t}^{(g')} \geqslant 65-x)}. \tag{4.16}$$

Note that 1_{surv} is a random variable, because it depends on $1_{(T_{y,t}^{(g')} \geqslant 65-x)}$, and on $1_F(\bar{t})$. The latter is likely to depend on death rates at time $\bar{t} = t + 65 - x$.

ii) For an insured aged $x < 65$ without a partner, the survivor annuity has no value, and so he will prefer a single life annuity, i.e., $1_{surv} = 0$.

iii) For a retiree ($x \geqslant 65$), 1_{surv} is realized. Specifically, $1_{surv} = 1_{surv,ret}$, where $1_{surv,ret} = 0$ if the participant *chose* a single life annuity at date $\bar{t} = t + 65 - x < t$, and $1_{surv,ret} = 1$ if the participant chose a joint and survivor annuity.

Combining choice behavior as described above with payments conditional on choice as given in (4.13) and (4.14) allows to write the present value of payments in a pension plan, L, as a combination of the present value of payments of a single life and a survivor annuity, L_{sl} and L_{surv}, as defined in (4.5) and (4.6). This yields the following result.

Proposition 4.5.1 *Consider a participant characterized by $((x,g),p,(y,g'))$ at time t, and let $\bar{t} = t + 65 - x$ denote the participant's retirement date. Then,*

$$L = \delta_1 \cdot L_{sl} + \delta_2 \cdot L_{surv},$$

with:

i) For an insured aged $x < 65$ in a JointLife plan,

$$\begin{aligned} \delta_1 &= 1 + \Big(1 - 1_F(\bar{t}) \cdot {}_{65-x}p_{y,t}^{(g')} \cdot 1_{(p=1)}\Big) \cdot w \cdot e(\bar{t}), \\ \delta_2 &= 1_F(\bar{t}) \cdot w \cdot 1_{\{p=1\}}; \end{aligned} \tag{4.17}$$

and, in a SingleLife plan,

$$
\begin{aligned}
\delta_1 &= 1 - 1_F(\bar{t}) \cdot {}_{65-x}p_{y,t}^{(g')} \cdot \frac{w \cdot e(\bar{t})}{1 + w \cdot e(\bar{t})} \cdot 1_{(p=1)}, \\
\delta_2 &= 1_F(\bar{t}) \cdot \frac{w}{1 + w \cdot e(\bar{t})} \cdot 1_{(p=1)}.
\end{aligned} \tag{4.18}
$$

ii) For a retiree, δ_1 is as in (4.17) and (4.18) with $1_F(\bar{t}) \cdot {}_{65-x}p_{y,t}^{(g')}$ replaced by $1_{pp,ret}$, and δ_2 as in (4.17) and (4.18) with $1_F(\bar{t})$ replaced by $1_{pp,ret}$.

The proof can be found in Appendix 4.B. For an insured who already reached retirement age at date t, i.e., $x \geqslant 65$, the nominal rights δ_1 and δ_2 are deterministic, because both the conversion rate $e(\bar{t})$, as well as the choice indicator, $1_{pp,ret}$, were realized at date $\bar{t} < t$. In contrast, the nominal insured rights for an insured who did not yet reach retirement age are affected by systematic longevity risk. They depend on the conversion rate at date $\bar{t} = t + 65 - x > t$, which is unknown at date t. In addition, for a participant with a partner, they depend on whether the partner is still alive at date $\bar{t}$ (i.e., they depend on the survival probability ${}_{65-x}p_{y,t}^{(g')}$), and, if so, on the couple's preference ordering, at date $\bar{t}$, between a single life annuity and a joint and survivor annuity (i.e., on the choice indicator $1_F(\bar{t})$). Both these factors depend on death rates up to time $\bar{t}$.

4.6 Longevity risk

In this section we use the results of the previous three sections to quantify the effect of longevity risk on the liabilities of a *JointLife* plan and of a *SingleLife* plan, respectively. Proposition 4.5.1 suggests that the two key factors that are likely to affect a plan's sensitivity to longevity risk are:

i) *The product mix*, i.e., the ratio of insured rights for the single life annuity over insured rights for the survivor annuity.

ii) *Dependence between the nominal insured rights, after exchange, for the single life and the survivor annuity, δ_1* and δ_2, *respectively, and the corresponding duration of the payments, L_{sl}* and L_{surv}. This dependence occurs because nominal rights depend on the conversion rate $e(\bar{t})$, and duration of payments depends on the remaining lifetimes of the participant and/or his partner. Both the conversion rate and the remaining lifetimes depend on uncertain future death rates.

It can easily be verified that the effect of product mix, i.e., $\frac{\delta_2}{\delta_1+\delta_2}$, is identical in the two types of plans. This occurs because in both types of plans, the ratio of survivor annuity rights over single life annuity rights for a participant that chooses a joint and survivor annuity equals w. In contrast, dependence effects are likely to depend strongly on the type of pension plan, and, within each plan, on the participant's choice between a single life annuity and a joint and survivor annuity at retirement date.

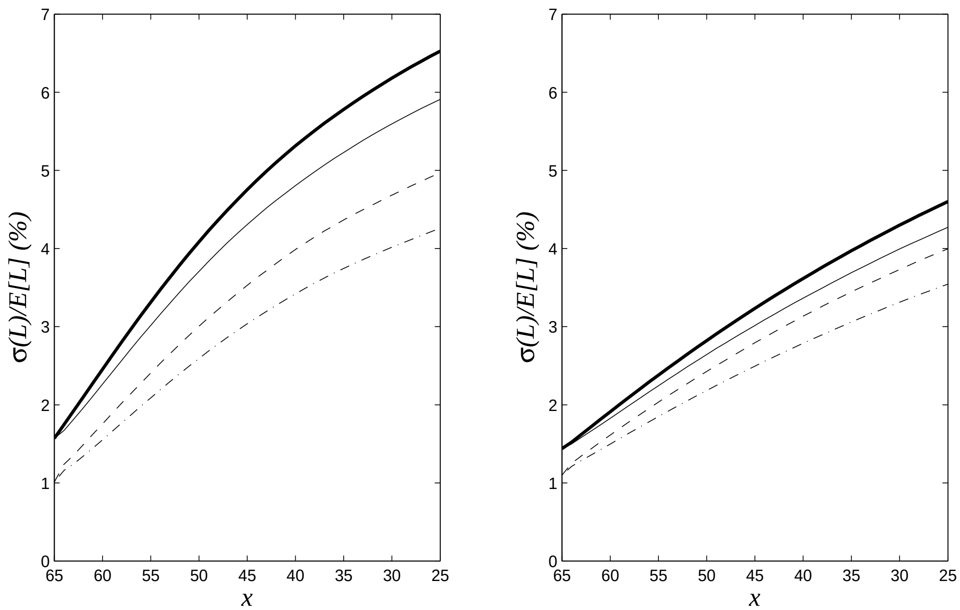
In the following subsections, we analyze these effects in detail. In Subsection 4.6.1, we analyze the effect of the participant's preference between the two types of annuities on longevity risk by comparing settings in which the participant will prefer a single life annuity to settings in which the participant will prefer a joint and survivor annuity. In Subsection 4.6.2, we quantify longevity risk taking into account that the participant's choice may depend on future survival probabilities. In all numerical illustrations, partners are assumed to be of different gender, i.e., $g = m$ implies $g' = f$, and vice versa; the partner of a male insured is three years younger ($y = x - 3$), and the partner of a female insured is three years older ($y = x + 3$). The interest rate is set to 4%, i.e., $r = 0.04$.

4.6.1 Effect of choice at retirement date

In this section we investigate how the effect of systematic longevity risk in the two types of plans depends on the participant's choice at retirement date. Because a participant without a partner will always choose a single life annuity, it is sufficient to distinguish two types of participants: First, participants who will choose a single life annuity at retirement date, i.e., participants without a partner ($p = 0$), or participants with a partner who prefer to waive survivor annuity rights ($p = 1$ and $1_F(\bar{t}) = 0$); second, participants with a partner who prefer a joint and survivor annuity in case both are alive at retirement date ($p = 1$ and $1_F(\bar{t}) = 1$).

Figure 4.6.1 displays $\sigma(L)/\mathbb{E}[L]$ as a function of the age of the participant, for the two pension plans, and for two types of insureds: an insured who will choose a single life annuity at retirement date, and a couple that will choose a joint and survivor annuity in case they are both alive. The left panel is for males and the right panel for females.

Figure 4.6.1: $\sigma(L)/\mathbb{E}[L]$ **as a function of age** x



This figure displays $\sigma(L)/\mathbb{E}[L]$ as a function of age x, for $p = 0$, or $p = 1$ and $1_F(\bar{t}) = 0$ (bold and solid lines), and $p = 1$ and $1_F(\bar{t}) = 1$ (dashed and dotted lines), for a participant in a *JointLife* plan (solid and dotted lines) and for a participant in a *SingleLife* plan (bold and dashed lines). Left panel: males; right panel: females. In each case, $w = 2/3$.

We see that in both types of plans longevity risk is substantially lower for a participant who prefers a joint and survivor annuity than for a participant who prefers a single life annuity. Moreover, for both choices, longevity risk is substantially lower in a *JointLife* plan than in a *SingleLife* plan. This is summarized in the following graph.

Importance of longevity risk

joint annuity		joint annuity		single life annuity		single life annuity
in	$\prec$	in	$\prec$	in	$\prec$	in
JointLife plan		*SingleLife* plan		*JointLife* plan		*SingleLife* plan

Below, we discuss these two results in detail.

<u>The effect of the participant's choice ($1_F(\bar{t})$)</u>

For every age x, for both genders, and for both types of pension plans, longevity risk risk is higher for a participant who will choose a single life annuity at retirement date (i.e., $p = 0$ or $p = 1$ and $1_F(\bar{t}) = 0$).

These results are driven by the fact that, for all ages and for both genders, single life and survivor annuity payments are negatively correlated. This occurs because a higher than expected increase in life expectancy of the participant implies a higher than expected value of the single life annuity, but a lower than expected value of the survivor annuity. The latter is due to the fact that the increase in life expectancy of the participant delays survivor annuity payments, so that they are more heavily discounted. Although increased life expectancy of the partner can lead to a longer duration of survivor annuity payments, this effect is dominated by the cost reducing effect of the delay in onset of the payments. For male insureds, the correlation between L_{sl} and L_{surv} decreases from -0.6 at age 65 to -0.7 at age 25. For female insureds, it decreases from -0.8 at age 65 to -0.87 at age 25. Note that the differences between the two types of insureds are larger for men than for women. This occurs because the hedge potential of survivor annuity rights is larger for males than for females.

The effect of the type of pension plan

For every age x, for both genders, and for the two choices, longevity risk is higher in the SingleLife plan than in the JointLife plan.

This result is driven by the fact that the exchange option induces dependence between the level of payments for the single life and the survivor annuity (δ_1 and δ_2), and the corresponding duration of payments (L_{sl} and L_{surv}). Depending on the type of plan and the participant's choice, δ_1 and δ_2 depend on the conversion rate $e(\bar{t})$. For almost all ages $x < 65$, and for both genders, the correlation between $e(\bar{t})$ and L_{sl} is negative, i.e., $\rho(e(\bar{t}), L_{sl}) < 0$, and the correlation between $e(\bar{t})$ and L_{surv} is positive, i.e., $\rho(e(\bar{t}), L_{surv}) > 0$.[9] Combined with Proposition 4.5.1, this yields the following results for the correlation between the level of payments and the duration

[9]Figure 4.A.1 in Appendix 4.A displays the correlation between $e(\bar{t})$ and L_{sl} as well as the correlation between $e(\bar{t})$ and L_{surv} for both males and for females, and provides the corresponding intuition.

of payments in the two plans.

	JointLife		*SingleLife*	
	$\rho(\delta_1, L_{sl})$	$\rho(\delta_2, L_{surv})$	$\rho(\delta_1, L_{sl})$	$\rho(\delta_2, L_{surv})$
$p = 0$	$-$	0	0	0
$p = 1, 1_F(\bar{t}) = 0$	$-$	0	0	0
$p = 1, 1_F(\bar{t}) = 1$	$-$	0	$+$	$-$

In case of a *JointLife* plan, the correlation between δ_1 and L_{sl} is negative in all three cases, which implies that a higher than expected increase in the duration of single life annuity payments (due to higher than expected reduction in mortality rates) is partly mitigated by a higher than expected reduction in the level of the annuity payment, δ_1. This hedge effect is not present in case of a *SingleLife* plan. There, levels and durations are uncorrelated, except for the coupled participant who prefers a joint and survivor annuity. For this participant, the effect of higher than expected duration of single life annuity payments is combined with higher than expected nominal payments. Although higher than expected duration of survivor annuity payments is partly mitigated by lower than expected nominal payments, the former effect is dominant.

4.6.2 Effect of longevity risk in choice at retirement date

The previous subsection shows that the effect of systematic longevity risk on each plan's liabilities depends strongly on the participant's choice at retirement date. However, the choice itself can also be affected by longevity risk, because it is likely to depend on death rates at the time the choice is made. In this subsection, we consider a participant with a partner characterized by $((x, g), 1, (y, g'))$ at date t, and we model the probability distribution of the random variable $1_F(\bar{t})$ that indicates whether the couple, in case they are both alive at time $\bar{t} = t + 65 - x$, will prefer a joint and survivor annuity ($1_F(\bar{t}) = 1$), or a single life annuity ($1_F(\bar{t}) = 0$).

Changes in death probabilities can have two countervailing effects on the preferred choice:

i) We know from Section 4.3 that the probability distribution of the conversion rate shifts downward over time. For any given death probabilities, a decrease in the conversion rate, ceteris paribus, makes a single life annuity less attractive in both plans. In a *JointLife* plan, a lower conversion rate implies that less single life annuity rights are received in return for survivor annuity rights of

w. In a *SingleLife* plan, it implies that less single life annuity rights need to be exchanged in return for survivor annuity rights.

ii) For any given conversion rate, a decrease in death probabilities of the participant, ceteris paribus, can reduce the attractiveness of a joint and survivor annuity because it induces a delay in the onset of survivor annuity payments.

In this section we set up a simple household utility model to quantify the aggregate effect. We consider a setting where a couple receives social security income of a in case they are both alive, and of a' in case either the insured or the partner is deceased. In addition, depending on the type of plan, the participant has accrued the right to receive a joint and survivor annuity (in a *JointLife* plan), or a single life annuity (in a *SingleLife* plan).[10] Let C denote the participant's accrued rights for the single life annuity in a *JointLife* plan. Then, the accrued rights for the survivor annuity equal $w \cdot C$. To guarantee actuarial equivalence of the accrued rights in both plans, we let the accrued rights for a single life annuity in a *SingleLife* plan equal $\big(1 + w \cdot \mathbb{E}[e(\bar{t})]\big) \cdot C$.

At retirement date (i.e., at time $\bar{t} = t + 65 - x$), the couple can, in both plans, choose between a single life and a joint and survivor annuity, i.e., they can exchange single life annuity rights for survivor annuity rights, or vice versa. We denote $1_f \in \{0, 1\}$ for a given choice, where $1_f = 0$ if the couple chooses a single life annuity, and $1_f = 1$ if they choose a joint and survivor annuity. We denote $1_F(\bar{t})$ for the couple's optimal choice.

In order to determine the couple's optimal choice, we first determine how the couple's income in future periods depends on their choice. Let $I(\tau|1_f)$ denote the household income (from social security and pension plan) in period $\bar{t}+\tau$, for a given choice $1_f \in \{0, 1\}$. Because at time $\bar{t}$, the participant is 65 years old, and the partner

[10] To avoid overloaded notation, we focus on the case where one partner has accrued pension rights in a pension plan, and the other has not. The model can easily be extended to allow for the case where both partners have accrued pension rights.

is $65+y-x$ years old, it follows from (4.10) and (4.12) that:[11]

$$
\begin{aligned}
I(\tau|1_f) \quad &= \widetilde{C}\cdot\left[1+(1-1_f)\cdot w\cdot e(\bar{t})\right]+a, && \text{if } 1_{T^{(g)}_{65,\bar{t}}\geq\tau}=1 \text{ and } 1_{T^{(g\prime)}_{y+65-x,\bar{t}}\geq\tau}=1;,\\
&= \widetilde{C}\cdot 1_f\cdot w+a', && \text{if } 1_{T^{(g)}_{65,\bar{t}}\geq\tau}=0 \text{ and } 1_{T^{(g\prime)}_{y+65-x,\bar{t}}\geq\tau}=1;\\
&= \widetilde{C}\cdot\left[1+(1-1_f)\cdot w\cdot e(\bar{t})\right]+a', && \text{if } 1_{T^{(g)}_{65,\bar{t}}\geq\tau}=1 \text{ and } 1_{T^{(g\prime)}_{y+65-x,\bar{t}}\geq\tau}=0;\\
&= 0, && \text{if } 1_{T^{(g)}_{65,\bar{t}}\geq\tau}=0 \text{ and } 1_{T^{(g\prime)}_{y+65-x,\bar{t}}\geq\tau}=0,
\end{aligned}
\tag{4.19}
$$

where $\widetilde{C}=C$ in case of a *JointLife* plan, and $\widetilde{C}=C\cdot\frac{1+w\cdot\mathbb{E}[e(\bar{t})]}{1+w\cdot e(\bar{t})}$ in case of a *SingleLife* plan.

It now remains to specify the couple's utility function. We assume an intertemporally separable, expected lifetime utility function with equal weights for both partners, where the utility derived from household income level x depends on whether both partners are alive, or only one partner is alive. Specifically,

$$
\begin{aligned}
u(x) \quad &= u_c(x) \quad && = 2\cdot\tfrac{1}{1-\beta}\cdot\left((1+\lambda)\cdot\tfrac{x}{2}\right)^{1-\beta}, && \text{if both are alive, and } \beta\neq 1;\\
& && = 2\cdot\ln\left((1+\lambda)\cdot\tfrac{x}{2}\right) && \text{if both are alive, and } \beta=1;\\
&= u_s(x) && = \tfrac{1}{1-\beta}\cdot x^{1-\beta}, && \text{if only one is alive, and } \beta\neq 1;\\
& && = \ln(x), && \text{if only one is alive, and } \beta=1,
\end{aligned}
$$

where β denotes the risk aversion coefficient of the household, and λ denotes the complementarity parameter (see, e.g., Brown and Poterba 2000). Then, conditional on given death probabilities, the couple's lifetime expected utility from a given choice 1_f is given by:

$$
\begin{aligned}
U(1_f) &:= \sum_{\tau=0}^{45}(1+\rho)^{-\tau}\cdot\mathbb{E}[u(I(\tau|1_f))\mid\mathcal{F}_\infty]\\
&= \sum_{\tau=0}^{45}(1+\rho)^{-\tau}\Big[u_c\left(\widetilde{C}\cdot[1+(1-1_f)\cdot w\cdot e]+a\right)\cdot\ {}_\tau p^{(g)}_{65,\bar{t}}\cdot{}_\tau p^{(g')}_{y+65-x,\bar{t}}\\
&\quad + u_s\left(\widetilde{C}\cdot 1_f\cdot w+a'\right)\cdot(1-{}_\tau\, p^{(g)}_{65,\bar{t}})\cdot\ {}_\tau p^{(g')}_{y+65-x,\bar{t}}\\
&\quad + u_s\left(\widetilde{C}\cdot[1+(1-1_f)\cdot w\cdot e]+a'\right)\cdot\ {}_\tau p^{(g)}_{65,\bar{t}}\cdot(1-\ {}_\tau p^{(g')}_{y+65-x,\bar{t}})\Big],
\end{aligned}
\tag{4.20}
$$

where ρ denotes the time preference parameter. Now, given death probabilities at time $\bar{t}$, the household will choose $1_f\in\{0,1\}$ such as to maximize the expected

[11] Conditional on both being alive at time $\bar{t}$, it follows from (4.9) that $1_{surv}=1_f$.

utility, i.e., they solve:

$$1_F(\bar{t}) = \arg \max_{1_f \in \{0,1\}} \mathbb{E}\left[U(1_f) \mid \mathcal{F}_{\bar{t}}\right]. \tag{4.21}$$

To investigate the effect of longevity risk on the couple's optimal choice, we simulated 15.000 scenarios for future death probabilities, and determined $1_F(\bar{t})$ in each scenario for all ages, both genders, both types of plans, for $\rho = r = 0.04$, $a = 1365, a' = 997$, and various combinations of the parameters C, β, and λ.[12] We found that:

i) *Male expected utility maximizers almost surely choose for a joint and survivor annuity.* This can be understood as follows. In both plans, a risk neutral male insured (i.e., $\beta = \lambda = 0$) will prefer a joint and survivor annuity at date $\bar{t}$ if, in expectation, it yields a higher present value of payoffs, i.e., if $\mathbb{E}\left[L_{sl}|\mathcal{F}_{\bar{t}}\right] + w\mathbb{E}\left[L_{surv}|\mathcal{F}_{\bar{t}}\right] > \left(1 + we(\bar{t})\right)\mathbb{E}\left[L_{sl}|\mathcal{F}_{\bar{t}}\right]$, or, equivalently,

$$\frac{\mathbb{E}\left[L_{surv}|\mathcal{F}_{\bar{t}}\right]}{\mathbb{E}[L_{sl}|\mathcal{F}_{\bar{t}}]} > e(\bar{t}) = \frac{\mathbb{E}\left[L_{surv,n,\bar{t}}|\mathcal{F}_{\bar{t}}\right]}{\mathbb{E}[L_{sl,n,\bar{t}}|\mathcal{F}_{\bar{t}}]}. \tag{4.22}$$

Now, the fact that with high probability, males will have higher death rates than females implies that the gender specific single life annuity value $\mathbb{E}[L_{sl}|\mathcal{F}_{\bar{t}}]$ almost surely is strictly lower than the gender-neutral value $\mathbb{E}[L_{sl,n,\bar{t}}|\mathcal{F}_{\bar{t}}]$. For the same reason, $\mathbb{E}[L_{surv}|\mathcal{F}_{\bar{t}}]$ is almost surely strictly higher than the gender-neutral value $\mathbb{E}[L_{surv,n,\bar{t}}|\mathcal{F}_{\bar{t}}]$. Thus, (4.22) is almost surely satisfied. The actuarial unfairness at individual level of the gender- and age-neutral conversion rate makes a joint and survivor annuity particularly attractive to a risk neutral male. Risk aversion (i.e., $\beta > 0$) even increases the attractiveness of a joint and survivor annuity, because it yields a smoother consumption pattern compared to a single life annuity. Although a higher value of λ or a lower value of C increases the attractiveness of a single life annuity, we find that for reasonable values of λ and C, the effect of the (unfairness of the) gender- and age-neutral conversion rate is dominant.

ii) *For the same reasons as described above, female risk neutral insureds would almost surely prefer a single life annuity.* In contrast to males, however, their preferred choice does vary nontrivially with the degree of risk aversion. For sufficiently low values of β, women will almost surely choose for a single life

[12]The values of a and a' are based on the monthly Dutch social security benefits for couples and singles.

annuity, due to the actuarial unfairness of the conversion rate. For sufficiently high degrees of risk aversion, however, this effect is dominated by the concern to receive a smooth income pattern, and so a joint and survivor annuity is preferred. For intermediate values, the choice depends on survival probabilities, and thus varies with age. For example, when $\lambda = 0.5$, and $\beta = 3$, the probability that a woman will choose for a joint and survivor annuity at retirement date decreases from 1 for a current retiree to approximately 0.46 for a currently 25 year old, in both plans.

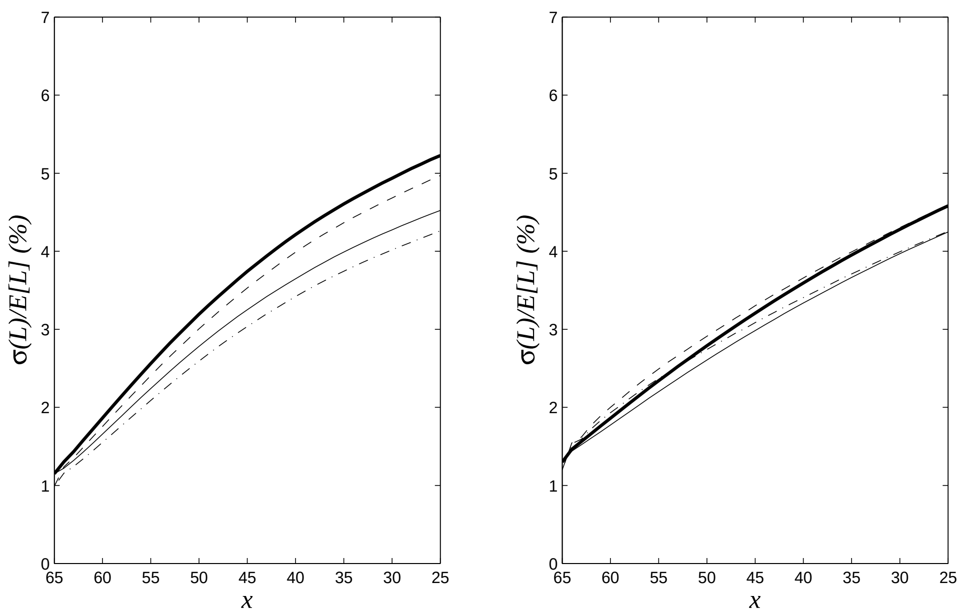
The fact that male expected utility maximizers almost always choose a joint and survivor annuity suggests that actual choice is unlikely to be fully explained by such rational behavior. Indeed, Johnson, Uccello, and Goldwyn (2005) find that 28% of male retirees with a partner choose a single life annuity. A plausible explanation could be that individuals use subjective death probabilities in making economic decisions, see, e.g., Salm (2006) and Gan, Gong, Hurd, and McFadden (2004). Self-reported survival probabilities are typically subject to measurement error and prone to focal point response (see, e.g., Hamermesh 1985, Hurd and McGarry 1995, and Gan, Hurd, and McFadden 2005). Because a joint and survivor annuity becomes more attractive if the the insured believes that his partner has significantly higher life expectancy, subjective survival probabilities can have a strong impact on the choice. Therefore, following Salm (2006), we model subjective survival probabilities ($\widetilde{p}_{x,t}^{(g)}$ and $\widetilde{p}_{y,t}^{(g')}$) as:

$$\begin{aligned}
\widetilde{p}_{x,t}^{(g)} &= \max\left\{0, 1 - \xi \cdot \left(1 - p_{x,t}^{(g)}\right)\right\}, \\
\widetilde{p}_{y,t}^{(g')} &= \max\left\{0, 1 - \xi' \cdot \left(1 - p_{x,t}^{(g')}\right)\right\},
\end{aligned}$$

where ξ and ξ' denote the *pessimism factor* of the participant and the partner, respectively. A value of ξ (ξ') higher than 1 indicates that the participant (partner) underestimates his survival probability. To investigate the effect of subjective survival probabilities on a couple's choice at retirement date, we consider risk neutral participants, i.e., $\beta = \lambda = 0$, who use subjective survival probabilities. We assume that ξ and ξ' are independent LogNormal random variables, and, based on Salm (2006), we let $\mathbb{E}[\xi] = \mathbb{E}[\xi'] = 1.041$, $\sigma[\xi] = \sigma[\xi'] = 1.029$. We then find that, for current as well as future retirees, approximately 72% of males and 24% of females choose a joint and survivor annuity.

We conclude by investigating whether and how choice behavior affects longevity risk in the two plans. In Figure 4.6.2, we display $\sigma(L)/\mathbb{E}[L]$ as a function of the age of the participant, for male and female participants with a partner, for the two pension plans, and with $1_F(\bar{t})$ determined by (4.21) for the two types of behavior: risk averse expected utility maximizers with $C = 1000, a = 1365, a' = 997, \lambda = 0.5$, $P(\beta = i) = 0.2$, for $i = 1, 2, \ldots, 5$, and $\rho = 0.04$; and risk neutral subjective decision makers with $\rho = 0.04$. The left panel is for males and the right panel for females.

Figure 4.6.2: $\sigma(L)/\mathbb{E}[L]$ **as a function of age**



This figure displays $\sigma(L)/\mathbb{E}[L]$ as a function of age, for risk averse expected utility maximizers with $C = 1000, a = 1365, a' = 997, \lambda = 0.5$, $P(\beta = i) = 0.2$, for $i = 1, 2, \ldots, 5$, and $\rho = 0.04$ (dashed and dotted lines), and risk neutral subjective decision makers with $\rho = 0.04$ (solid and bold lines), in a *JointLife* plan (solid and dotted lines) and in a *SingleLife* plan (bold and dashed lines). Left panel: males, right panel: females.

As discussed in Subsection 4.6.1, the difference in longevity risk between the two plans is strongly driven by correlation between the nominal insured rights δ_1 and δ_2, and the corresponding duration of payments, L_{sl} and L_{surv}. The fact that the choice variable $1_F(\bar{t})$ depends on survival rates affects these correlations. Compared to the case where $1_F(\bar{t})$ is independent of future death rates, the correlations between nominal rights and durations weakly increase, and, therefore, longevity risk weakly increases. However, because the correlations are less positive (more negative) in a *JointLife* plan than in a *SingleLife* plan, it is still the case that the *JointLife* plan is significantly less affected by longevity risk than the *SingleLife* plan.

4.7 Conclusions and further research

This chapter investigates the effect of longevity risk on pension plans that, at retirement date, allow the participant to choose between a single life annuity and a joint and survivor annuity. We show that these plans are affected by longevity risk in two ways. First, the participant's choice at retirement date affects the ratio of survivor benefits over single life benefits, and, thus, affects the hedge potential that arises form combining these benefits. Second, for insureds who are not yet retired, the conversion rate is a random variable that depends on future realizations of death rates. We show that conversion rate uncertainty significantly affects both the participant and the pension fund. For the participant, depending on his preferences and the type of pension plan, it induces uncertainty in the level of the single life annuity rights and/or survivor annuity rights. For the pension fund, it induces dependence between the level of the payments and the duration of the payments. We find that a pension plan where participants accrue both single life and survivor annuity rights, and are allowed to exchange their survivor annuity rights for additional single life annuity rights, is significantly less affected by longevity risk than the alternative plan where participants accrue only single life annuity rights, and can exchange part of those rights for survivor annuity rights.

Let us finally indicate some interesting directions for future research. First, we have ignored interest rate risk. Even in the presence of interest rate risk, we expect the correlation between the level and the duration of payments for single life and survivor annuities, respectively, to be less positive (more negative) in a *JointLife* plan than in a *SingleLife* plan. Therefore, we do not expect that interest rate risk will affect the relative importance of longevity risk in the two plans. However, it could be interesting to quantify the relative importance of longevity risk and interest rate risk in each plan. Second, we have used a relatively simple model for a couple's choice between the two types of annuities at retirement date, e.g., ignoring optimal savings behavior and (adverse) selection issues. Selection effects due to, e.g., heterogeneity in survival rates may affect the plan's sensitivity to longevity risk. If the two types of plans coexist and individuals can choose the type of plan they enter, such selection effects could affect the relative importance of longevity risk in the two types of plans.

4.A Figure 4.A.1

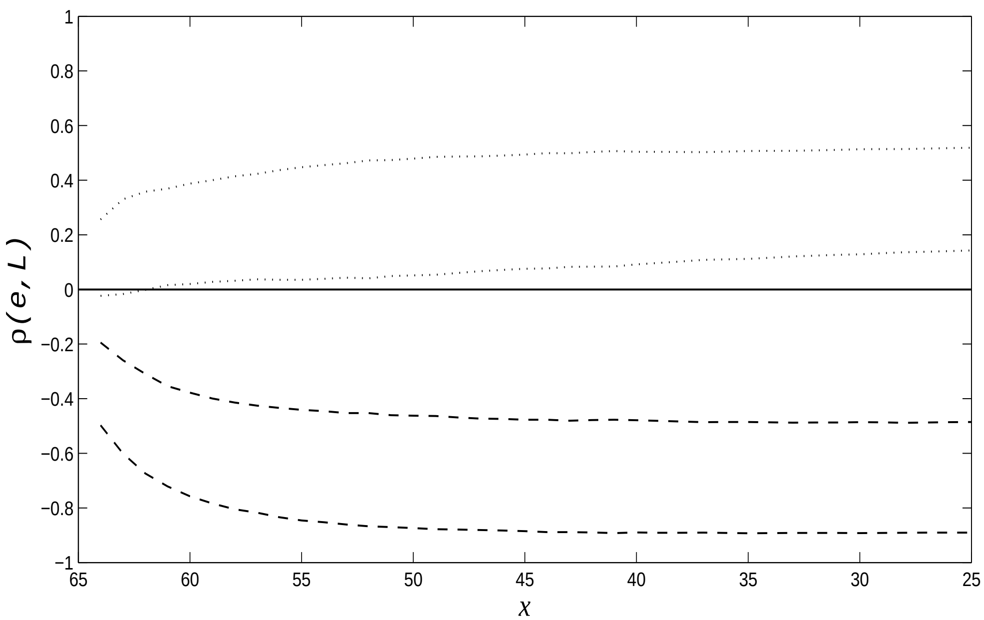
Remember that $e(\bar{t}) = \mathbb{E}\left[L_{surv,n,\bar{t}}|\mathcal{F}_{\bar{t}}\right]/\mathbb{E}[L_{sl,n,\bar{t}}|\mathcal{F}_{\bar{t}}]$. Now, $\rho(e(\bar{t}), L_{sl}) < 0$ and $\rho(e(\bar{t}), L_{surv}) > 0$, because

i) the gender- and age-neutral *expected* value of a single life annuity at date $\bar{t}$, $\mathbb{E}\left[L_{sl,n,\bar{t}}|\mathcal{F}_{\bar{t}}\right]$, is positively correlated with the date-t gender- and age specific liability L_{sl};

ii) $\mathbb{E}\left[L_{surv,n,\bar{t}}|\mathcal{F}_{\bar{t}}\right]$ and L_{surv}, are positively correlated;

iii) $\mathbb{E}\left[L_{sl,n,\bar{t}}|\mathcal{F}_{\bar{t}}\right]$ and L_{surv}, as well as $\mathbb{E}\left[L_{surv,n,\bar{t}}|\mathcal{F}_{\bar{t}}\right]$ and L_{sl}, are negatively correlated.

Figure 4.A.1: $\rho(e(\bar{t}), L_{sl})$ **as a function of** x



This figure displays $\rho(e(\bar{t}), L_{sl})$ as a function of x (dashed lines) for males (lower graph) and for females (upper graph), and $\rho(e(\bar{t}), L_{surv})$ as a function of x (dotted lines) for males (upper graph) and for females (lower graph).

4.B Proof of Proposition 4.5.1

i) Consider an insured aged $x < 65$ with a partner ($p = 1$). Then, in case of a *JointLife* plan, it follows from (4.13) that

$$\begin{aligned}\widetilde{L}(\tau) &= 0, && \text{for } \tau < 65 - x;\\ &= \left(1 + w \cdot e\left(\overline{t}\right) \cdot (1 - 1_{surv})\right) \cdot \widetilde{L}_{sl}(\tau) + w \cdot 1_{surv} \cdot \widetilde{L}_{surv}(\tau) && \text{for } \tau \geqslant 65 - x.\end{aligned} \tag{4.23}$$

Take any $\tau \geqslant 65-x$. Then, given (4.16), and since, conditional on $\mathcal{F}_\infty$, the random variables $T^{(g)}_{x,t}$, $T^{(g')}_{y,t}$, $1_F\left(\overline{t}\right)$, and $e\left(\overline{t}\right)$ are independent, and since $\mathbb{E}[e\left(\overline{t}\right)|\mathcal{F}_\infty] = e\left(\overline{t}\right)$, and $\mathbb{E}[1_F\left(\overline{t}\right)|\mathcal{F}_\infty] = 1_F\left(\overline{t}\right)$, it follows that:

$$\begin{aligned}&\mathbb{E}\left[\left(1 + w \cdot e\left(\overline{t}\right) \cdot (1 - 1_{surv})\right) \cdot \widetilde{L}_{sl}(\tau)|\mathcal{F}_\infty\right]\\ =\ &\mathbb{E}\left[\left(1 + w \cdot e\left(\overline{t}\right) \cdot \left(1 - 1_F\left(\overline{t}\right) \cdot 1_{(T^{(g')}_{y,t} \geqslant 65-x)}\right)\right) \cdot 1_{(T^{(g)}_{x,t} \geqslant \tau)}|\mathcal{F}_\infty\right]\\ =\ &\left(1 + w \cdot e\left(\overline{t}\right) \cdot \left(1 - \mathbb{E}\left[1_F\left(\overline{t}\right) \cdot 1_{(T^{(g')}_{y,t} \geqslant 65-x)}|\mathcal{F}_\infty\right]\right)\right) \cdot \mathbb{E}[1_{(T^{(g)}_{x,t} \geqslant \tau)}|\mathcal{F}_\infty]\\ =\ &\left(1 + w \cdot e\left(\overline{t}\right) \cdot \left(1 - 1_F\left(\overline{t}\right) \cdot {}_{65-x}p^{(g')}_{y,t}\right)\right) \cdot \mathbb{E}[\widetilde{L}_{sl}(\tau)|\mathcal{F}_\infty].\end{aligned}$$

Similarly,

$$\begin{aligned}\mathbb{E}\left[1_{surv} \cdot \widetilde{L}_{surv}(\tau)|\mathcal{F}_\infty\right] &= \mathbb{E}\left[1_{(T^{(g')}_{y,t} \geqslant 65-x)} \cdot 1_F\left(\overline{t}\right) \cdot \widetilde{L}_{surv}(\tau)|\mathcal{F}_\infty\right]\\ &= \mathbb{E}\left[1_{(65-x \leqslant T^{(g)}_{x,t} < \tau)} 1_{(T^{(g')}_{y,t} \geqslant \tau)} 1_{(T^{(g')}_{y,t} \geqslant 65-x)} \cdot 1_F\left(\overline{t}\right)|\mathcal{F}_\infty\right]\\ &= \mathbb{E}\left[1_{(65-x \leqslant T^{(g)}_{x,t} < \tau)} 1_{(T^{(g')}_{y,t} \geqslant \tau)}|\mathcal{F}_\infty\right] \cdot \mathbb{E}\left[1_F\left(\overline{t}\right)|\mathcal{F}_\infty\right]\\ &= 1_F\left(\overline{t}\right) \cdot \mathbb{E}[\widetilde{L}_{surv}(\tau)|\mathcal{F}_\infty].\end{aligned} \tag{4.24}$$

Thus, for all $\tau \geqslant 65 - x$, (4.23) implies

$$\begin{aligned}\mathbb{E}[\widetilde{L}(\tau)|\mathcal{F}_\infty] =\ &\left(1 + w \cdot e\left(\overline{t}\right)\left(1 - 1_F\left(\overline{t}\right) \cdot {}_{65-x}\, p^{(g')}_{y,t}\right)\right) \cdot \mathbb{E}[\widetilde{L}_{sl}(\tau)|\mathcal{F}_\infty]\\ &+ w \cdot 1_F\left(\overline{t}\right) \cdot \mathbb{E}[\widetilde{L}_{surv}(\tau)|\mathcal{F}_\infty].\end{aligned} \tag{4.25}$$

For $\tau < 65 - x$, it holds that $\widetilde{L}(\tau) = 0$. Therefore, it follows from (4.4), and (4.25), that

$$\begin{aligned}\sum_{\tau=0}^{110-x} \mathbb{E}\left[\widetilde{L}(\tau)|\mathcal{F}_\infty\right] \frac{1}{(1+r)^\tau} =\ &\left(1 + e\left(\overline{t}\right)\left(1 - 1_F\left(\overline{t}\right) \cdot {}_{65-x}\, p^{(g')}_{y,t}\right)\right) \cdot\\ &\sum_{\tau=65-x}^{110-x} \mathbb{E}\left[\widetilde{L}_{sl}(\tau)|\mathcal{F}_\infty\right] \frac{1}{(1+r)^\tau}\\ &+ w \cdot 1_F\left(\overline{t}\right) \sum_{\tau=65-x}^{110-x} \mathbb{E}[\widetilde{L}_{surv}(\tau)|\mathcal{F}_\infty] \frac{1}{(1+r)^\tau}.\end{aligned}$$

Now, (4.17) follows immediately from (4.5) and (4.6).

The proof for the *SingleLife* plan is similar.

ii) Follows immediately from (4.4), (4.13), and (4.14), and the fact that $1_{surv} = 0$, and $\mathbb{E}[e\left(\bar{t}\right)|\mathcal{F}_\infty] = e\left(\bar{t}\right)$.

iii) Follows immediately from (4.4), (4.13), and (4.14), and the fact that $1_{surv} = 1_{surv,ret}$, and $\mathbb{E}[e\left(\bar{t}\right)|\mathcal{F}_\infty] = e\left(\bar{t}\right)$.

4.C Forecasting future mortality

In this appendix we briefly describe the models used to quantify the systematic longevity risk affecting $p_{x,t}^{(g)}$. Let $\mu_{x,t}^{(g)}$ denote the force of mortality of a person with age x and gender g at time t, i.e., $\mu_{x,t}^{(g)} = \lim_{\Delta t \to 0} P\left(0 \leqslant T_{x,t}^{(g)} \leqslant \Delta t\right)/\Delta t$. We assume that for any integer age x, any gender g, and any time t, it holds that $\mu_{x+u,t}^{(g)} = \mu_{x,t}^{(g)}$, for all $u \in [0,1)$. Then, one can verify

$$p_{x,t}^{(g)} = \exp\left(-\mu_{x,t}^{(g)}\right).$$

Next, let $D_{x,t}^{(g)}$ denote the observed number of deaths in year t in a cohort with gender g and aged x at the beginning of year t, and let $E_{x,t}^{(g)}$ denote the number of person years during year t in a cohort with gender g and aged x at the beginning of year t, the so-called exposure. Then, under appropriate regularity conditions, as discussed by Gerber (1997), the Maximum Likelihood estimator for the force of mortality is given by $\widehat{\mu}_{x,t}^{(g)} = D_{x,t}^{(g)}/E_{x,t}^{(g)}$. We assume that there is no (sampling) risk involved in this relationship, so that the systematic longevity risk is fully captured by the risk in $D_{x,t}^{(g)}/E_{x,t}^{(g)}$. We use the models proposed by Lee and Carter (1992), Brouhns, Denuit, and Vermunt (2002a), and Cossette et al. (2007) to quantify the systematic longevity risk in $D_{x,t}^{(g)}/E_{x,t}^{(g)}$.

The model by Lee and Carter (1992) is given by

$$\log\left(D_{x,t}^{(g)}/E_{x,t}^{(g)}\right) = a_x^{(g)} + b_x^{(g)} k_t^{(g)} + \epsilon_{x,t}^{(g)}, \tag{4.26}$$

where $k_t^{(g)}$ is an index of the level of mortality, $a_x^{(g)}$ is an age-specific constant describing the general pattern of mortality by age, $b_x^{(g)}$ is an age-specific constant describing the relative speed of the change in mortality by age, and where $\epsilon_{x,t}^{(g)}$ represents the measurement error, assumed to satisfy $\epsilon_{x,t}^{(g)} \mid \mathcal{K}_t \sim N\left(0, \sigma_{x,g}^2\right)$, conditional on $\mathcal{K}_t = \left\{k_\tau^{(g)} \mid g \in \{m,f\}, \tau = t, t-1, ...\right\}$. Moreover, we assume that the $\epsilon_{x,t}^{(g)}$ are independent for different x and g, conditional on $\mathcal{K}_t$.

To model the process for $\left(k_t^{(m)}, k_t^{(f)}\right)$ over time, we use an ARIMA(0,1,1) model

$$k_t^m = k_{t-1}^m + c^m + \theta^m e_{t-1}^m + e_t^m, \tag{4.27}$$
$$k_t^f = k_{t-1}^f + c^f + \theta^f e_{t-1}^f + \rho e_t^m + e_t^f, \tag{4.28}$$

where c^g is the gender g specific drift term which indicates the average annual change of k_t^g, θ^g is the gender specific moving average coefficient, and e_t^g is the gender specific innovation such that

$$\begin{pmatrix} e_t^m \\ e_t^f \end{pmatrix} \mid \mathcal{K}_{t-1} \sim N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \sigma_m^2 & 0 \\ 0 & \sigma_f^2 \end{pmatrix} \right).$$

The parameter ρ captures the correlation between k_t^m and k_t^f over time.

In case of the model by Brouhns, Denuit, and Vermunt (2002a), the age and gender specific numbers of deaths are modeled by a Poisson process,

$$D_{x,t}^{(g)} \mid \widetilde{\mathcal{K}}_t \sim \text{Poisson}\left(E_{x,t}^{(g)} e^{a_x^{(g)} + b_x^{(g)} k_t^{(g)}}\right), \tag{4.29}$$

with $\widetilde{\mathcal{K}}_t = \mathcal{K}_t \bigcup \left\{ E_{x,\tau}^{(g)} \mid g \in \{m, f\}, \text{all } x, \tau = t, t-1, ... \right\}$ We assume that the $D_{x,t}^{(g)}$ are independent for different x and g, conditional on $\widetilde{\mathcal{K}}_t$. The process for $\left(k_t^{(m)}, k_t^{(f)}\right)$ is modeled as in case of the Lee and Carter (1992)-model, i.e., via equations (5.29)–(5.30).

As third model, we consider Cossette et al. (2007). These authors model the age specific numbers of deaths $D_{x,t}^{(g)}$ via the Binomial Gumbel process,

$$D_{x,t}^{(g)} \mid \widetilde{\mathcal{K}}_t \sim \text{Bin}\left(E_{x,t}^{(g)}, 1 - exp\left(-e^{a_x^{(g)} + b_x^{(g)} k_t^{(g)}}\right)\right), \tag{4.30}$$

where we again assume that the $D_{x,t}^{(g)}$ are independent for different x and g, conditional on $\widetilde{\mathcal{K}}_t$, and where we model the process for $\left(k_t^{(m)}, k_t^{(f)}\right)$ via equations (5.29)–(5.30).

To forecast the future mortality rates, we use the current time (defined in the appendix as time T) mortality table and a reduction factor. In this way the mortality rates are set such that the estimation error in the last year of the mortality data is zero, so that we avoid a jump-off bias in the forecasts. Let $q_{x,T}^{(g)} = 1 - p_{x,T}^{(g)}$ be based on the last year of mortality data. Then we forecast $q_{x,T+s}^{(g)}$ as follows

$$\widehat{q}_{x,T+s}^{(g)} = q_{x,T}^{(g)} \times \widehat{RF}_{x,T,s}^{(g)}. \tag{4.31}$$

The reduction factor is given by

$$\widehat{RF}_{x,T,s}^{(g)} = e^{\widehat{b}_x^{(g)} \times (\widehat{k}_{T+s}^{(g)} - \widehat{k}_T^{(g)})}, \tag{4.32}$$

where $\widehat{b}_x^{(g)}$ and $\widehat{k}_T^{(g)}$ denote the (model specific) estimated $b_x^{(g)}$ and $k_T^{(g)}$, respectively, and where $\widehat{k}_{T+s}^{(g)}$ denotes the $s \geqslant 1$ periods ahead forecast. For the latter we use

$$\begin{pmatrix} \widehat{k}_{T+s}^{m} \\ \widehat{k}_{T+s}^{f} \end{pmatrix} \mid \widehat{\mathcal{K}}_{T+s-1} \sim N\left(\begin{pmatrix} \widehat{k}_{T+s-1}^{m} + \widehat{c}^{m} \\ \widehat{k}_{T+s-1}^{f} + \widehat{c}^{m} \end{pmatrix}, \begin{pmatrix} \widehat{\sigma}_m^2 & \widehat{\rho}^2 \widehat{\sigma}_m^2 \\ \widehat{\rho}^2 \widehat{\sigma}_m^2 & \widehat{\rho}^2 \widehat{\sigma}_m^2 + \widehat{\sigma}_f^2 \end{pmatrix} \right), \tag{4.33}$$

with $\widehat{\mathcal{K}}_T = \mathcal{K}_T$, and $\widehat{\mathcal{K}}_{T+s} = \widehat{\mathcal{K}}_{T+s-1} \bigcup \left\{ \widehat{k}_{T+s}^{g}, \widehat{k}_{T+s}^{f} \right\}$, for $s = 1, 2, 3, ...$, employing (model specific) estimates.

In order to avoid localized age induced anomalies in $\widehat{b}_x^{(g)}$ in the three models, we follow Renshaw and Haberman (2003). These authors proposed to smooth the age specific estimated parameters $\widehat{b}_x^{(g)}$ using cubic B-splines, with internal knots,

$$\zeta_0^{(g)} + \zeta_1^{(g)} x + \zeta_2^{(g)} x^2 + \zeta_3^{(g)} x^3 + \sum_{j=1}^{r} \zeta_{3+j}^{(g)} (x - x_j)_+^3, \tag{4.34}$$

where $(x - x_j)_+^3 = (x - x_j)^3$, in case $x - x_j > 0$, and zero otherwise. As internal knots we use $x_1 = 9.5$, $x_2 = 20.5$, $x_3 = 50.5$, $x_4 = 60.5$, and $x_r = x_5 = 80.5$. The cubic B-splines are fitted to the (model specific) estimated $\widehat{b}_x^{(g)}$ using the method of least squares.

The model-specific parameters are estimated imposing the required normalizations and using the estimation techniques as described in the corresponding papers. For the Lee and Carter (1992)-model we first estimate the parameters $a_x^{(g)}$, $b_x^{(g)}$, and $k_t^{(g)}$ using a singular value decomposition (SVD). Secondly, for all $t \leqslant T$ and $g \in \{m, f\}$, we re-estimate $k_t^{(g)}$ such that the estimated number of deaths using the estimates of $a_x^{(g)}$ and $b_x^{(g)}$ in equation (1) (with $\epsilon_{x,t}^{(g)} = 0$) equals the observed number of deaths. These re-estimated $k_t^{(g)}$, $t \leqslant T$, are used to estimate the process for $\left(k_t^{(m)}, k_t^{(f)}\right)$ using equations (5.29)–(5.30). For the Brouhns, Denuit, and Vermunt (2002a)-model and the Cossette et al. (2007)-model we use the iterative procedure proposed by Goodman (1979) to obtain the Maximum Likelihood estimates, where the criterium to stop the procedure is a very small (i.e., 10^{-10}) increase of the log-likelihood.

Age, gender, and time specific numbers of death and exposed to death are obtained from the Human Mortality Database.[13] In our case $x \in \{0, 1, 2, ..., 99, 100^{+}\}$,

[13] See www.mortality.org.

with 100^+ the age group of people aged 100 years or more. We use the time period 1977–2004, so that $T = 2004$. This time period minimizes the statistic proposed by Booth et al. (2002b) to test the hypothesis that the age components ($b_x^{(g)}$) are invariant over time. We use this selection, since mortality experience in the industrialized world seems to suggest a substantial age-time interaction in the twentieth century.

The parameter estimates relevant for the quantification of the systematic longevity risk are plotted in Figure 5.B.1 (the $\widehat{b}_x^{(g)}$) and presented in Table I (the parameter estimates of equations (5.29)–(5.30)).

Table 4.C.1: **Estimation results**

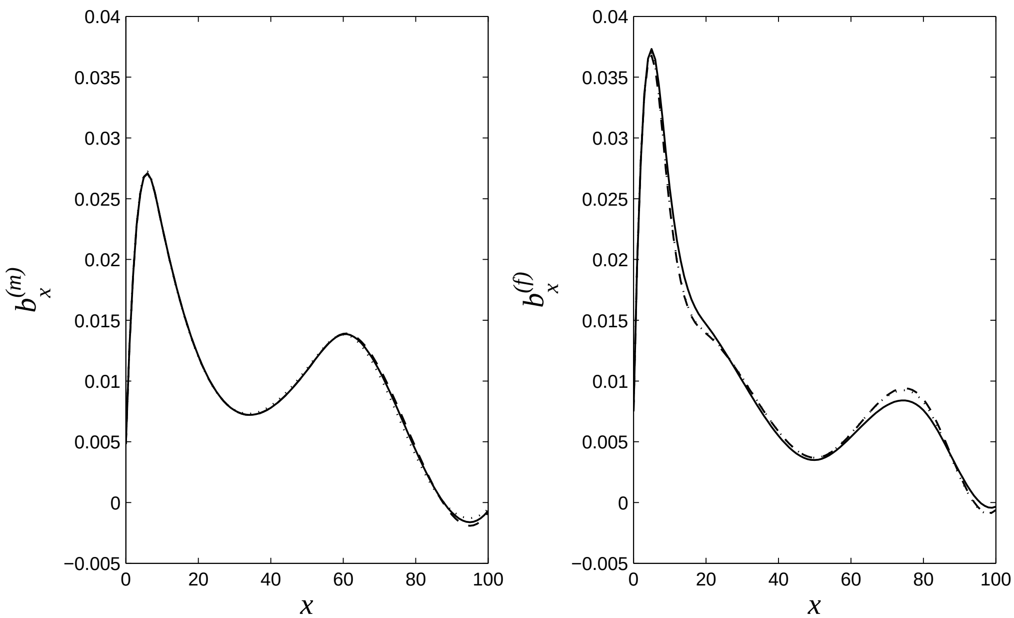
Model	g	$c^{(g)}$	$\theta^{(g)}$	σ_g	ρ
Lee-Carter	m	-1.6923	-0.4196	1.0628	0.4545
	f	-1.2902	-0.6056	1.3475	
Brouhns, Denuit, and Vermunt	m	-1.60771	-0.3386	0.7318	0.5873
	f	-1.2572	-0.5286	0.9491	
Cossette et al.	m	-1.6063	-0.3449	0.7270	0.5944
	f	-1.2535	-0.5273	0.9321	

Parameter estimates of equations (5.29)–(5.30)). Lee-Carter: Lee and Carter (1992)-model; Brouhns, Denuit, and Vermunt: Brouhns, Denuit, and Vermunt (2002a)-model; Cossette et al.: Cossette et al. (2007)-model.

The estimation results for the different models are quite comparable, with the estimation results of the Lee and Carter (1992) model slightly deviating from the other two models. At younger ages males and females are more sensitive to changes in the time trend, while also males around 60 and females around 77 show an increased sensitivity, see Figure 5.B.1. In terms of the $k_t^{(g)}$-processes, we find that the male drift term is more negative than the female drift term, but in case of females the first order moving average term is more negative. The risk in the female process (given by $\widehat{\rho}^2\widehat{\sigma}_m^2 + \widehat{\sigma}_f^2$) is also substantial higher than in case of the males (given by $\widehat{\sigma}_m^2$). Finally, there is a substantial correlation between the male and female process.

We include three sources of systematic longevity risk: process risk, parameter risk, and model risk. First, using (4.31) and (4.32), given a specific model and given the corresponding model specific estimates, there is *process risk* due to fact that future values of $\widehat{k}_{T+s}^{(g)}$ are risky, see equation (4.33). Next, given a specific model, the

Figure 4.C.1: **Estimated $b_x^{(g)}$ after smoothing using cubic B-splines**



This figure displays the estimated $b_x^{(g)}$ after smoothing using cubic B-splines. Left panel: $g = m$; right panel: $g = f$. The solid curve corresponds to the Lee and Carter (1992)-model; the dashed curve corresponds to the Brouhns, Denuit, and Vermunt (2002a)-model, and the dotted curve corresponds to the Cossette et al. (2007)-model.

forecasts (4.31)–(4.33) are based on model specific estimates, and these estimates are sensitive to estimation inaccuracy. The corresponding risk is referred to as *parameter risk*. Finally, different models might be used to calculate the forecasts (4.31)–(4.33). Assuming that some prior distribution is used to do the forecast calculations, there is also *model risk*.

To quantify the systematic longevity risk, we proceed as follows. Given the initial data

$$\left\{ \left(D_{x,t}^{(g)}, E_{x,t}^{(g)} \right) \mid x \in \{0, 1, 2, ..., 99, 100^+\}, g \in \{m, f\}, t \in \{1977, \ldots, T = 2004\} \right\},$$

the following steps are taken.

1) For each of the three models, the parameters $\widehat{a}_x^{(g)}$, $\widehat{b}_x^{(g)}$, and $\widehat{k}_t^{(g)}$ are estimated

and the corresponding residuals $r_{x,t}^{(g)}$ are computed.[14] Let R_t be the matrix with components $r_{x,t}^{(g)}$, for $g \in \{m, f\}, x \in \{0, ..., 100^+\}$.

2) Next, for each model, we generate $B = 5000$ replications $\overline{R}_t(b)$, $b = 1, ..., B$, of the residual matrix R_t, by sampling with replacement.[15] Using these residual matrices, the corresponding (model specific) bootstrapped numbers of death $\overline{D}_{x,t}^{(g)}(b)$, $b = 1, ..., B$, are determined.[16]

3) Given the bootstrapped numbers of death $\overline{D}_{x,t}^{(g)}(b)$, we compute the (model specific) bootstrap estimates $\widehat{a}_x^{(g)}(b)$, $\widehat{b}_x^{(g)}(b)$ $\widehat{k}_t^{(g)}(b)$, $b = 1, ..., B$, using the described estimation techniques.

4) Given the bootstrap estimates $\widehat{a}_x^{(g)}(b)$, $\widehat{b}_x^{(g)}(b)$, $\widehat{k}_t^{(g)}(b)$, we generate $\widehat{k}_{T+s}^m(b)$ and $\widehat{k}_{T+s}^f(b)$, using the model specific version of (4.33), for $s = 1, ..., 85$ and $b = 1, ..., B$. This allows us to calculate the corresponding $p_{x,T+s}^{(g)}(b)$ via (4.31) and (4.32), resulting in $\mathcal{F}_t(b)$ for appropriate t.

5) Finally, for some quantity of interest $F = F(\mathcal{F}_t)$, we calculate the (model specific) bootstrap values $F(b) \equiv F(\mathcal{F}_t(b))$, for $b = 1, ..., B$, for each of the three models. On the basis of the distribution of all bootstrap values of $F(b)$, merged over the three models, we are able to quantify the systematic longevity risk.

[14] In case of the Brouhns, Denuit, and Vermunt (2002a)- and the Cossette et al. (2007)-model, we calculated the deviance residuals.

[15] This corresponds to the residual bootstrap percentile interval-method of Efron and Tibshirani (1998). See also Koissi, Shapiro, and Högnäs (2006).

[16] In case of deviance residuals, this requires the use of the inverse relationship between numbers of death and deviance residuals.

Chapter 5

Hedge effects in a portfolio of life insurance products with investment risk

This chapter is based on Stevens, De Waegenaere and Melenberg (2010b).

5.1 Introduction

Our goal in this chapter is to quantify longevity risk in portfolios of life insurance products, taking into account the potential effect of investment risk on the impact of longevity risk. Specifically, our focus is on potential interactions between liability mix effects and asset mix effects.

Existing literature suggests that uncertainty regarding the future development of human life expectancy potentially imposes significant risk on pension funds and insurers (see, for example, Olivieri and Pitacco, 2001; Brouhns, Denuit, and Vermunt, 2002; Cossette, Delwarde, Denuit, Guillot, and Marceau, 2007; Dowd, Cairns, and Blake, 2006; Hári, De Waegenaere, Melenberg, and Nijman, 2008). Existing literature also shows that the natural hedge potential that arises from combining life annuities and death benefits may be substantial (see, for example, Cox and Lin, 2007; Wang, Huang, Yang, and Tsai, 2010; Tsai, Wang, and Tzeng, 2010). These analyses quantify longevity risk in annuity portfolios by determining its effect on the probability distribution of the *present value* of all future payments, for a given, deterministic, and constant term structure of interest rates. A drawback of this approach is that it does not allow to take into account the possible interaction between longevity risk and financial risk, i.e., it is a "liability-only" approach. Hári et al. (2008) quantify longevity risk in portfolios of single life annuities in the presence of financial risk by determining its effect on the volatility of the *funding ratio*. The funding ratio is defined as the ratio of the value of the assets over the value of the liabilities. They find that financial risk can significantly affect the impact of longevity risk on funding ratio volatility. However, a drawback of a funding ratio approach is that it requires specifying the probability distribution of the value of the liabilities at a future date. Determining the value of longevity-linked liabilities is still a contentious issue. Although in recent years there has been considerable interest in developing pricing models for longevity-linked assets and liabilities (see, for example, Blake and Burrows, 2001; Dahl, 2004; Lin and Cox, 2005; Denuit, Devolder, Goderniaux, 2007; Bauer, Boerger, and Russ, 2010), the lack of liquidity for trade in longevity-linked assets and/or liabilities makes it very difficult to calibrate these models. As long as this remains the case, it is unclear to what extent a funding ratio approach accurately reflects the effect of longevity risk.

Our goal in this chapter is threefold. First, we quantify the impact of longevity risk in portfolios of life insurance products, taking into account potential interactions between financial risk and longevity risk. To avoid making any assumptions regarding the value at which longevity-linked liabilities can be sold, we quantify risk by means of the probability of ruin in a run-off approach. Specifically, for any given investment strategy, we determine the minimal required buffer (i.e., the asset value in excess of the best estimate value of the liabilities), such that the probability that the insurer or pension fund will be able to pay all future liabilities is sufficiently high (see, for example, Olivieri and Pitacco, 2003). The size of the buffer will be affected by longevity risk, which arises due to uncertain deviations in the future liability payments from their current best estimates, and by financial risk, which arises due to uncertainty in future returns on assets. Part of the financial risk arises due to uncertain returns on the assets needed to cover *unexpected* deviations of the liabilities from their expected values, and, therefore, cannot be fully hedged. We find that the effect of this unhedgeable financial risk on the required solvency buffer depends significantly on the type of liability. This suggests important interactions between financial risk and longevity risk.

Second, we quantify the effect of unhedgeable financial risk on the natural hedge potential, i.e., the risk reduction, that arises from combining liabilities with different sensitivities to longevity risk. Whereas financial risk is typically hedgeable for a deterministic stream of liabilities, the unhedgeable financial risk arises from the uncertainty in the stream of future payments. Life insurers and pension funds often hold several types of longevity-linked liabilities, such as single life annuities, last survivor annuities, and death benefit insurance.[1] Because the payments of these different life insurance products typically have different sensitivities to changes in mortality rates, insurers with a "diversified" portfolio of liabilities may be less sensitive to longevity risk.[2] The existing literature on such liability mix effects focuses on the natural hedge potential, i.e., risk reduction, of death benefits in portfolios of life annuities, and uses a liability-only approach to quantify the risk reduction.[3] We quantify the effect of investment risk on the natural hedge potential from combining life insurance products with different sensitivities to longevity risk. We find, for example, that ignoring unhedgeable investment risk may lead to significant overestimation of the hedge potential from death benefits in portfolios of single life annuities. The extent to which the hedge potential is overestimated depends nontrivially on the liability mix.

Third, we quantify the effect of potential interactions between liability mix effects and asset mix effects on the risk reduction from investing in survivor swaps. Because the payments of survivor swaps are based on actual survival of a reference population, they may be used to partially hedge longevity risk. Existing literature shows that the hedge potential can be affected by basis risk, i.e., residual risk due to differences in characteristics of the insured population and the reference population (see, for example, Dowd, Cairns, and Blake, 2006). In this chapter we show that, in addition to basis risk, the hedge potential of survivor swaps also depends nontrivially on both the asset mix and the liability mix. Depending on the liability mix, the hedge potential of survivor swaps may either increase or decrease when investment risk is higher.

The chapter is organized as follows. In Section 5.2 we define the life insurance

[1] Many defined benefit pension funds offer both old-age pension insurance and partner pension insurance. The latter consists of a survivor annuity that yields periodic payments if the partner of the insured person is alive and the insured person has passed away. The Retirement Equity Act of 1984 (REA) amended the Employee Retirement Income Security Act of 1974 (ERISA) to introduce mandatory spousal rights in pension plans.

[2] Cox and Lin (2007) show empirically that a life insurer who has 95% of its business in annuities and 5% in death benefits prices its annuities on average 3% higher than an insurer who has 50% of its business in annuities and 50% of its business in death benefits. This indicates that insurers with death benefit liabilities have a competitive advantage.

[3] Wang et al. (2010) and Tsai et al. (2010) quantify the natural hedge potential of death benefits in portfolios of life annuities, and determine the optimal liability mix.

liabilities that we consider, and discuss how they are affected by longevity risk. Section 5.3 gives a formal definition of the risk measure. Section 5.4 shows how investment risk affects the impact of longevity risk in single life annuities, survivor annuities, and death benefits, respectively. In Section 5.5, we quantify the effect of the interaction between liability mix effects and asset mix effects. Section 5.6 deals with the effect of liability and asset mix on the hedge potential of survivor swaps. Section 5.7 concludes.

5.2 Life insurance liabilities and longevity risk

In this section we introduce the life insurance liabilities that we consider, and discuss how they are affected by systematic and non-systematic longevity risk.

In addition to traditional old-age pensions, which take the form of a single life annuity, pension funds and insurers typically also offer other types of life insurance products, such as partner pensions and death benefits. A partner pension consists of a survivor annuity. It provides the partner of a deceased participant with a life long annuity payment. The death benefit consists of a single payment at the moment the insured person dies. Formally, we consider the following three types of liabilities:

(i) A *single life annuity*, which yields a nominal yearly payment of 1, with a last payment in the year the insured person dies;

(ii) A *survivor annuity*, which yields a nominal yearly payment of 1 in every year that the spouse outlives the insured person;

(iii) A *death benefit*, which yields a nominal single payment of 1 in the year that the participant dies.

We let $\mathcal{P} = \{sl, surv, db\}$ denote the set of life insurance products, and we denote a product by $p \in \mathcal{P}$, where $p = sl$ refers to a singe life annuity, $p = surv$ refers to a survivor annuity, and $p = db$ refers to a death benefit. These liabilities consist of (a stream of) payments in future periods. Because in any future period, the level of the payment depends on whether the insured person is alive, and, in case of survivor annuities, whether the partner is alive, the net cash outflow of these life insurance products is affected by two types of longevity risk:

- *non-systematic longevity risk:* conditional on given survival probabilities, whether an individual survives an additional year is a random variables;

- *systematic longevity risk:* the survival probabilities for future dates are uncertain.

While non-systematic longevity risk is diversifiable (i.e., the risk becomes negligible when portfolio size is large, see, for example, Olivieri and Pitacco, 2001), this is not the case for systematic longevity risk. Therefore, throughout the chapter we assume that portfolios are large enough for non-systematic longevity risk to be negligible, and focus on the impact of systematic longevity risk. Because survival rates depend significantly on age and gender, we characterize an insured/participant by a vector $(\overline{x}, \overline{g})$, where

$$\begin{array}{lllllll} \overline{x} & = & x, & \overline{g} & = & g, & \text{if } p \in \{sl, db\}, \\ \overline{x} & = & (x, y), & \overline{g} & = & (g, g'), & \text{if } p = surv, \end{array}$$

where x denotes the age of the insured, $g \in \{m, f\}$ denotes the gender of the insured, and, in case of survivor annuities, y denotes the age of the partner, and $g' \in \{m, f\}$ denotes her/his gender. Then, for any given year t, the liability payments in a future year $t + \tau$, $\tau \geq 0$ for a single life annuity, a survivor annuity, and a death benefit insurance for an individual characterized by $(\overline{x}, \overline{g})$ in year t, are given by (see, for example, Gerber 1997):[4]

$$\begin{aligned} \widetilde{L}_{p,\tau,t}(\overline{x}, \overline{g}) &= {}_{\tau}p^{(g)}_{x,t}, && \text{for } p = sl \text{ (singe life annuity)}, \\ &= \left(1 - {}_{\tau}p^{(g)}_{x,t}\right) \cdot {}_{\tau}p^{(g')}_{y,t}, && \text{for } p = surv \text{ (survivor annuity)}, \\ &= {}_{\tau-1}p^{(g)}_{x,t} - {}_{\tau}p^{(g)}_{x,t}, && \text{for } p = db \text{ (death benefit)}, \end{aligned} \tag{5.1}$$

where

- $p^{(g)}_{x+s,t+s}$ for $s >= 0$ denote the realized one-year survival probabilities of the cohort aged x in year t, as defined in (2.5);

- ${}_{\tau}p^{(g)}_{x,t} = p^{(g)}_{x,t} \cdot p^{(g)}_{x+1,t+1} \cdots p^{(g)}_{x+\tau-1,t+\tau-1}$ denotes the realized τ-years survival probability of the cohort aged x in year t.

[4] Existing literature shows that there exists dependence between the remaining lifetimes of a participant and his (her) partner at micro-level, e.g., due to the fact that partners have similar lifestyles, or that the passing away of a partner affects the surviving relative's quality of life. Because our focus in this chapter is on systematic longevity risk, we ignore this dependence and assume that the remaining lifetimes of the spouses, conditional on the survival probabilities, are independent.

We consider a given and fixed date t, and quantify the risk in the liabilities in a run-off approach in which there are no new entrants in the portfolio, and no premiums are paid after date t. Without loss of generality, we let $t = 0$ and suppress the dependence on t unless it is required for clarity. Because our focus is on the interaction between liability mix and asset mix effects, we will consider portfolios consisting of several products, with varying weights, and with insureds with varying characteristics. Specifically, let $\mathcal{I}$ denote the set of insureds. The total payment in year τ is of the form

$$\widetilde{L}_\tau = \sum_{i\in\mathcal{I}} \sum_{p\in\mathcal{P}} \overline{\delta}_{i,p} \cdot \widetilde{L}_{p,\tau}(\overline{x}_i, \overline{g}_i), \tag{5.2}$$

where $\overline{\delta}_{i,p}$ denotes the insured right of insured i for pension product p. Throughout the chapter, we denote BEL for the current (i.e., date-0) best estimate value of the liabilities, which is defined as the market value of the expected liabilities, i.e.,

$$BEL = \sum_{\tau=1}^{\infty} \mathbb{E}\left[\widetilde{L}_\tau\right] \cdot P^{(\tau)}, \tag{5.3}$$

where $P^{(\tau)}$ denotes the current market value of a zero-coupon bond with maturity τ. In Subsection 5.3.2 we discus the calculation of he expectation in (5.3)

5.3 Quantifying risk

In this section we discuss how we quantify risk in portfolios that are sensitive to both longevity risk and financial risk. In Subsection 5.3.1 we formally define the risk measure. In Subsection 5.3.2, we provide a brief discussion of the models according to which the risk in the death rates, interest rates, and asset returns are generated. A complete description of these models can be found in Appendices 5.A and 5.B.

5.3.1 Risk measure

We quantify risk in portfolios of life insurance products by determining, for any given investment strategy, the minimal initial asset value such that the probability that the *terminal asset value* is positive is sufficiently large. The terminal asset value is defined as the remaining asset value after the last payment has been made. Without loss of generality, we express the initial asset value A_0 as the best estimate value of the liabilities, BEL, plus a buffer that is a percentage of the best estimate value, i.e.,

$$A_0 = (1+c) \cdot BEL. \tag{5.4}$$

Then, for a given $\varepsilon > 0$, we determine the minimum value of the buffer percentage c such that:

$$\mathbb{P}\left(A_T < 0 \mid A_0 = (1+c) \cdot BEL\right) \leqslant \varepsilon, \tag{5.5}$$

where T denotes the last period in which a payment needs to be made, and A_T denotes the corresponding terminal asset value.

The minimal required buffer percentage c depends on the probability distribution of the terminal asset value, A_T, which in turn depends on the initial asset value A_0, the liability payments, $\widetilde{L}_\tau$ (as defined in (5.2)), and the investment strategy. Specifically, the asset dynamics is given by:

$$A_\tau = (1+r_\tau) \cdot A_{\tau-1} - \widetilde{L}_\tau, \qquad \tau = 1, \cdots, T,$$

where A_τ denotes the net asset value at the end of period τ, r_τ denotes the return on assets during period τ, and $\widetilde{L}_\tau$ denotes the liabilities paid at the end of period τ. Because we want to be able to distinguish between hedgeable and unhedgeable financial risk, we allow for the case where the insurer uses a different investment strategy for the best estimate value (BEL) and for the buffer ($c \cdot BEL$). Specifically, we define the following strategies.

Definition 5.3.1 *An investment strategy consists of:*

- *for every duration $\tau = 1, \cdots, T$: an asset mix for the best estimate value corresponding to duration τ (i.e., the amount $\mathbb{E}\left[\widetilde{L}_\tau\right] \cdot P^{(\tau)}$); the corresponding return in periods $s = 0, \cdots, \tau$ is denoted $r_s^{be,(\tau)}$;*
- *an asset mix for the buffer portfolio; the corresponding return in periods $\tau = 0, \cdots, T$ is denoted r_τ^{bu}.*

In every period τ, the accumulated value of the best estimate portfolio corresponding to duration τ is used to pay the liabilities in period τ; any shortage or excess is taken from, or reinvested in, the buffer portfolio.

Whereas the value of the buffer portfolio is affected by both longevity risk and investment risk, the value of the best estimate portfolio is only affected by investment risk. For example, when the buffer portfolio is invested in equity and the best estimate portfolio in zero-coupon bonds, a lower return on the assets, or a higher than expected realization of the liabilities, leads to a smaller proportion of assets invested in equity.

With the above defined investment strategy, we obtain the following result.

Proposition 5.3.2 *The minimum required buffer percentage to satisfy (5.5) is given by:*

$$c = \frac{Q_{1-\epsilon}\left(L\right)}{BEL} - 1, \tag{5.6}$$

where

$$L = BEL + \sum_{\tau=1}^{T}\left(\frac{\widetilde{L}_{\tau} - \mathbb{E}\left[\widetilde{L}_{\tau}\right]\cdot P^{(\tau)}\cdot\prod_{s=1}^{\tau}\left(1+r_s^{be,(\tau)}\right)}{\prod_{s=1}^{\tau}\left(1+r_s^{bu}\right)}\right), \tag{5.7}$$

and $Q_{1-\varepsilon}(L)$ denotes the $(1-\varepsilon)-$quantile of L.

Proof: The date-τ value of the best estimate portfolio corresponding to duration τ is given by $\mathbb{E}\left[\widetilde{L}_{\tau}\right]\cdot P^{(\tau)}\cdot\prod\limits_{s=0}^{\tau}\left(1+r_s^{be,(\tau)}\right)$. Combined with (5.3), this implies that the terminal asset value is given by:

$$\begin{aligned} A_T &= c\cdot BEL\prod_{\tau=1}^{T}(1+r_{\tau}^{bu})+\sum_{\tau=1}^{T}\left(\mathbb{E}\left[\widetilde{L}_{\tau}\right]P^{(\tau)}\prod_{s=1}^{\tau}(1+r_s^{be,(\tau)})-\widetilde{L}_{\tau}\right)\prod_{s=\tau}^{T}(1+r_s^{bu}) \\ &= \left[(1+c)\cdot BEL - L\right]\cdot\prod_{\tau=1}^{T}\left(1+r_{\tau}^{bu}\right), \end{aligned} \tag{5.8}$$

with L as defined in (5.7). Therefore, the terminal asset value A_T is nonnegative if

$$(1+c)\cdot BEL \geqslant L, \tag{5.9}$$

The result now follows immediately from (5.5). □

The above proposition shows that the required buffer percentage follows from determining the $1-\varepsilon$ quantile of the random variable L. The random variable L

can be interpreted as follows. Conditional on any given future asset returns (r_s^{bu} and $r_s^{be,(\tau)}$), and cash flows ($\widetilde{L}_\tau$ as defined in (5.2)), L represents the value of the assets needed at date 0 to pay all future liability payments. For the sake of intuition, consider for example the case where all assets yield a deterministic and constant annual return, i.e., $r_s^{bu} = r_s^{be,(\tau)} = r$ for some $r > 0$, and $P^{(\tau)} = 1/(1+r)^\tau$. Then it follows immediately from (5.3) and (5.7) that L simplifies to:

$$L = \sum_{\tau=1}^{T} \frac{\widetilde{L}_\tau}{(1+r)^\tau}, \tag{5.10}$$

i.e., L equals the discounted present value of all future liability payments. Thus, the standard approach in which longevity risk is quantified by determining its effect on the probability distribution of the present value of liabilities can be seen as a special case of our model. The more general case in (5.7), however, allows to take into account interactions between financial risk and longevity risk.

5.3.2 Modeling mortality rates and asset returns

To determine the minimum required buffer from (5.6), we simulate 15,000 scenarios for death rates and asset returns, and on the basis of these scenarios we calculate the $1-\varepsilon$ quantile of L. In this subsection we briefly describe the models we use to generate these scenarios.

To model asset returns, we use a Vasicek model for the term structure of interest rates, combined with a Geometric Brownian motion with time-varying drift for stock prices. We include both process risk (i.e., risk given estimated parameter values) and parameter risk (i.e., risk due to estimation inaccuracy). To estimate the parameters, we use the daily instantaneous short rate, the daily interest rate on a 10 years Dutch government bond, and the daily return on the Dutch stock index "AEX", obtained from Datastream. For a more detailed description of the models and the estimation technique, and for parameter estimates, we refer to Appendix 5.A. We use these models to generate 15000 scenarios for asset returns.

For the probability distribution of the future survival probabilities we include process risk, parameter risk, and model risk. To incorporate model risk, we estimate three classes of survival probability models, namely the Lee-Carter (1992) class of models, the Cairns-Blake-Dowd (2006) class of models, and the P-Splines model (Currie, Durbin, and Eilers; 2004). We generate 5,000 scenarios for future survival rates from each class of models: 5,000 scenarios from Lee-Carter (1992)-type models with three different specifications, namely the Lee-Carter (1992) model (1,666 scenarios), the Brouhns, Denuit, and Vermunt (2002) model (1667 scenarios), and the Cossete et al. (2007) model (1,667 scenarios); 5,000 scenarios from Cairns-Blake-Dowd (2006) models with four different specifications, allowing for a quadratic term in the age effect, and/or constant/diminishing age effects in the cohort effects (each specification 1,250 scenarios); and, 5,000 scenarios from the P-Splines model with one specification. To estimate the parameters in each model, we use age-, gender-, and time-specific number of deaths and exposures to death for the Netherlands, obtained from the Human Mortality Database. For a detailed description of the models and the estimation techniques, and for parameter estimates, we refer to Appendix 5.B.

5.4 Hedgeable and unhedgeable investment risk

In this section we investigate how investment risk affects the impact of longevity risk in single life annuities, survivor annuities, and death benefits, respectively. To do so, we decompose L from (5.7) into three components, i.e.,

$$L = BEL + L^{long} + L^{invest},$$

where

(i) $BEL = \sum_{\tau=1}^{\infty} \mathbb{E}\left[\widetilde{L}_\tau\right] \cdot P^{(\tau)}$ is the *deterministic component*. BEL is the market value of the *expected* liabilities. It represents the asset value that is needed on date 0 to pay all future expected liabilities, given that the expected liabilities are cash flow matched. Expected liabilities are cash flow matched iff for every duration τ, the amount $\mathbb{E}\left[\widetilde{L}_\tau\right] \cdot P^{(\tau)}$ is invested in (default-free) zero-coupon bonds with maturity τ.

(ii) L^{invest} is the *pure investment risk component*:

$$L^{invest} = \sum_{\tau=1}^{T} \frac{\mathbb{E}\left[\widetilde{L}_\tau\right] - \mathbb{E}\left[\widetilde{L}_\tau\right] \cdot P^{(\tau)} \cdot \prod_{s=1}^{\tau} \left(1 + r_s^{be,(\tau)}\right)}{\prod_{s=1}^{\tau} \left(1 + r_s^{bu}\right)}.$$

This component represents the asset value that, conditional on given asset returns, is needed on date 0 in addition to BEL to pay all future *expected* liabilities when expected liabilities are not cash flow matched. This component is affected by risk that arises due to uncertain deviations of the τ-years return on the best estimate portfolio ($\prod_{s=1}^{\tau}\left(1+r_s^{be,(\tau)}\right)$) from the cash-flow matching return ($\frac{1}{P^{(\tau)}}$).

(iii) L^{long} is the *longevity risk component*:

$$L^{long} = \sum_{\tau=1}^{T} \frac{\widetilde{L}_\tau - \mathbb{E}\left[\widetilde{L}_\tau\right]}{\prod_{s=1}^{\tau}\left(1+r_s^{bu}\right)}.$$

This component represents the asset value that, conditional on given asset returns, is needed at date 0 in addition to BEL and L^{invest} to pay all future *unexpected* liability payments (i.e., payments in excess of the expected value). This component is affected by two sources of longevity risk: *pure longevity risk* that arises due to deviations of the liabilities from their expected values ($\widetilde{L}_\tau - \mathbb{E}\left[\widetilde{L}_\tau\right]$), and *longevity induced investment risk* that arises due to uncertain returns on these deviations.

Thus, the present value variable L can be decomposed in a deterministic term that reflects the required asset value in absence of both longevity risk and financial risk (BEL), a term that reflects the required additional asset value in absence of longevity risk, but with financial risk (L^{invest}), and a term that reflects the required additional asset value due to longevity risk (L^{long}). Both L^{invest} and L^{long} are affected by investment risk, but only L^{long} is affected by longevity risk. Moreover, whereas L^{invest} reflects hedgeable risk (L^{invest} reduces to zero when the expected liabilities are cash flow matched), L^{long} reflects unhedgeable risk that arises due to uncertainty in the returns on assets required to cover unexpected liabilities.

In this chapter we investigate the hedge effects of different life insurance products. For the hedge effects in a portfolio of life insurance liabilities with investment risk we refer to:

- *hedgaeble investment risk* as the uncertainty in the pure investment risk component L^{invest};
- *unhedgaeble investment risk* as the uncertainty in the longevity risk component L^{long} arises from the uncertainty in r_τ^{bu};

- *longevity risk* as the uncertainty in the longevity risk component L^{long} arises from the uncertainty in $\widetilde{L}_\tau$;
- *natural hedge* potential as the risk reduction from combining different life insurance product in the longevity risk component L^{long}.

Note that the aim of the chapter is not to determine the optimal investment portfolio, but to investigate to what extent the hedge effects depend on the investment portfolio.

In the remainder of this section we illustrate the effect of financial risk on the impact of longevity risk by comparing the benchmark case, when L is defined as in (5.10), to the case where investment returns are uncertain. To quantify the effect of both hedgeable and unhedgeable interest rate risk, we compare two investment strategies. The first investment strategy is a "risky" one in which *all* assets are (re)invested in a risky portfolio. The second investment strategy is one in which the best estimate value is invested in bonds, and the buffer portfolio is invested in risky assets. Specifically, we consider the following two investment strategies:

- A *risky investment strategy* in which both the best estimate value BEL, and the buffer $c \cdot BEL$ are (re)invested in a portfolio that yields returns r_s^{bu}, i.e.,

  $$r_s^{be,(\tau)} = r_s^{bu}, \text{ for all } s = 0, \cdots, T, \text{ and } \tau = 1, \cdots, T.$$

  It then follows from Proposition 5.3.2 that the minimal required buffer percentage c is given by (5.6) with:

  $$L = \sum_{\tau=1}^{T} \frac{\widetilde{L}_\tau}{\prod_{s=1}^{\tau} (1 + r_s^{bu})}, \tag{5.11}$$

- A *best estimate hedge strategy* in which the best estimate value BEL is cash flow matched, i.e.,

  $$\prod_{s=1}^{\tau} \left(1 + r_s^{be,(\tau)}\right) = \frac{1}{P^{(\tau)}}, \quad \text{for all } s = 0, \cdots, T, \text{ and } \tau = 1, \cdots, T, \tag{5.12}$$

  and the buffer $c \cdot BEL$ is (re)invested in a portfolio that yields random returns r_s^{bu} in periods $s = 0, \cdots, T$. It then follows from Proposition 5.3.2 that the minimal required buffer percentage c is given by (5.6) with:

  $$L = BEL + \sum_{\tau=1}^{T} \frac{\widetilde{L}_\tau - \mathbb{E}\left[\widetilde{L}_\tau\right]}{\prod_{s=1}^{\tau} (1 + r_s^{bu})}. \tag{5.13}$$

This strategy eliminates hedgeable investment risk, i.e., $L^{invest} = 0$. Investment risk arises only due to uncertain deviations of $\widetilde{L}_\tau$ from its expectation $\mathbb{E}\left[\widetilde{L}_\tau\right]$. These (uncertain) deviations affect the value of the buffer portfolio, generating unhedgeable investment risk.

To investigate whether the impact of financial risk depends strongly on the type of liability, we consider two types of insured individuals, i.e., male insureds and female insureds aged $x = 65$, and three types of liabilities, i.e., single life annuities (i.e., $\widetilde{L}_\tau = \widetilde{L}_{sl,\tau}(x,g)$), survivor annuities (i.e., $\widetilde{L}_\tau = \widetilde{L}_{surv,\tau}(\overline{x},\overline{g})$), and death benefits (i.e., $\widetilde{L}_\tau = \widetilde{L}_{db,\tau}(x,g)$). In case of survivor annuities, the partner of a male insured is a female aged $y = 62$; the partner of a female insured is a male aged $y = 68$. Regarding asset returns, we consider the case where the buffer (and thus also the best estimate value in case of the risky strategy) is invested in one-year bonds.

We use the models described in the Appendix to simulate future investment returns and survival probabilities. We then use these simulated distributions to determine the minimum required buffer percentage c to reduce the probability of ruin to 2.5%, using (5.6) and (5.7) with $\varepsilon = 0.025$. Table 5.4.1 displays the minimal required buffer percentage c for the risky investment strategy (c^{risky}; second column), for the best estimate hedge strategy as defined in (5.12) (c^{BEh}; third column), and for the benchmark liability-only case with a deterministic return of $r = 4\%$ (c^{LO}; last column). Because it is intuitively clear that the effect of longevity risk as well as of financial risk on the required buffer may depend substantially on the duration of the liabilities, the first column displays the duration of the expected liabilities, which is given by:

$$\text{Duration} = \frac{\sum_{\tau=1}^{T} \tau \cdot P^{(\tau)} \cdot \mathbb{E}\left[\widetilde{L}_\tau\right]}{\sum_{\tau=1}^{T} P^{(\tau)} \cdot \mathbb{E}\left[\widetilde{L}_\tau\right]}.$$

Table 5.4.1 shows that the effect of investment risk on the minimal required buffer percentage depends heavily on the type of liability. First, compared to the liability-only approach (c^{LO}), the required buffer percentage under the risky investment strategy (c^{risky}) increases by a factor ranging from 2.5 (for female survivor annuities) to more than 9 (for female death benefits). These huge differences are partly due to the fact that under the naive investment strategy, there is a mismatch between the duration of the investments (one year) and the duration of the liabilities; this mismatch

Table 5.4.1: Minimal required buffer percentages

Product	Duration	c^{risky}	c^{BEh}	c^{LO}
Male single life annuity	8.2	26.5%	5.9%	4.9%
Female single life annuity	8.9	30.6%	7.4%	6.1%
Male survivor annuity	16.3	79.3%	23.5%	15.5%
Female survivor annuity	13.6	55.4%	38.2%	29.6%
Male death benefit	14.4	66.9%	10.6%	7.3%
Female death benefit	16.7	87.1%	16.5%	10.1%

induces significant reinvestment risk. Second, compared to the risky strategy, the best estimate hedge strategy (c^{BEh}) leads to significant reductions in the required buffer percentages. However, even with this conservative investment strategy in which all hedgeable investment risk is eliminated, the required buffer percentages are still significantly larger than under the liability-only approach. The extent to which the required buffer percentage is underestimated with the liability-only approach depends nontrivially on the type of liability. It varies from 20% for male single life annuities to 63% for female death benefits.

5.5 Effect of unhedgeable investment risk

The results of the previous section suggest that there are nontrivial interactions between longevity risk and investment risk; the effect of investment risk on the required buffer depends strongly on the type of liability. In this section we quantify the effect of these interactions on the impact of longevity risk in portfolios of life insurance products. To focus on longevity risk, we consider a best estimate hedge strategy as defined in (5.12). This ensures that all hedgeable investment risk is eliminated (i.e., L^{invest} is deterministic), and investment risk arises only due to uncertain returns on the buffer portfolio, which cannot be fully hedged because of the longevity uncertainty in the stream of the payments $\widetilde{L}_\tau - \mathbb{E}\left[\widetilde{L}_\tau\right]$.

Compared to the benchmark liability-only approach, taking into account investment risk implies that (comparing (5.13) to (5.10)):

(i) the expected liabilities are valued at market value, i.e., using a term structure of interest rates instead of a flat discount rate (i.e., BEL instead of $\sum_{\tau=1}^{T} \mathbb{E}\left[\widetilde{L}_\tau\right] / (1+r)^\tau$),

(ii) deviations from the expected value ($\widetilde{L}_\tau - \mathbb{E}\left[\widetilde{L}_\tau\right]$) are subject to uncertain returns (i.e., r_s^{bu} instead of r).

The first effect is deterministic, but the second is stochastic and can therefore nontrivially affect required buffer percentages. Specifically, uncertain buffer returns imply that L is affected by simultaneous deviations of the liabilities from their expected value (i.e., $\widetilde{L}_\tau - \mathbb{E}\left[\widetilde{L}_\tau\right] \neq 0$), and of the returns from the flat rate (i.e., $r_s^{bu} \neq r$). The effect of uncertain deviations of the liabilities from their expected values is aggravated (weakened) when these deviations are accompanied by lower (higher) than expected returns on the buffer portfolio. Therefore, changes in the liability mix will not only affect the "pure longevity risk" component, i.e., the risk given known future investment returns, but also the interactions between longevity risk and investment risk. Ignoring these interactions may lead to inaccurate quantification of the hedge potential that arises from combining different types of liabilities (for example, the natural hedge potential of death benefits).

In this section we investigate the effect of interactions between unhedgeable financial risk and longevity risk in portfolios with single life annuities, survivor annuities, and death benefits. To do so, we determine the buffer percentage c from (5.6) and (5.13) for various asset and liability mixes, and compare the results to the buffers resulting from a liability-only approach in (5.10) with $r = 4\%$. To quantify the impact of unhedgeable financial risk, we consider four different investment strategies for the buffer portfolio: 100% one-year zero-coupon bonds; 67% one-year zero-coupon bonds, 33% equity; 33% one-year zero-coupon bonds, 67% equity; and 100% equity. With regard to the liability mix, we consider portfolios that differ in terms of *gender mix* (ratios of male insured rights over total insured rights for each product) and in terms of *product mix* (ratios of insured rights for the different life insurance products) for each gender. Gender mix nontrivially affects the required buffer percentage because male and female mortality trends are not perfectly correlated. Product mix nontrivially affects the required buffer percentage because survivor annuity payments and single life annuity payments are negatively correlated. Therefore, we consider two types of insured individuals, male insureds and female insureds aged 65, who each may hold insured rights ($\overline{\delta}_{i,p}$, see (5.2)) for three different types of liabilities: single life annuities ($p = sl$), survivor annuities ($p = surv$), and death benefits ($p = db$). The partner of a male insured (if present) is aged 62; the partner of a female insured (if present) is aged 68. It is verified easily that the minimum required buffer percentage c is then given by (5.6) and (5.13) with:[5]

$$\begin{aligned}\widetilde{L}_\tau = (1-\gamma)\cdot\Big[\widetilde{L}_{sl,\tau}(65,f) + w_f\cdot\widetilde{L}_{surv,\tau}(65,68,f,m) + d_f\cdot\widetilde{L}_{db,\tau}(65,f)\Big] \\ +\gamma\cdot\Big[\widetilde{L}_{sl,\tau}(65,m) + w_m\cdot\widetilde{L}_{surv,\tau}(65,62,m,f) + d_m\cdot\widetilde{L}_{db,\tau}(65,m)\Big], \quad (5.14)\end{aligned}$$

where $\widetilde{L}_{sl,\tau}(\cdot)$, $\widetilde{L}_{surv,\tau}(\cdot)$, and $\widetilde{L}_{db,\tau}(\cdot)$ are as defined in (5.1), and where

- γ is the fraction of male single life annuities rights relative to the total single life annuities rights,
- w_g for $g \in \{m, f\}$ is the ratio of survivor annuity rights for gender g over single life annuities rights for gender g, and,
- d_g for $g \in \{m, f\}$ is the ratio of death benefit rights for gender g over single life annuities rights for gender g.

In Subsection 5.5.1 we investigate interactions between longevity risk and investment risk in portfolios of single life and survivor annuities ($d_g = 0$). In Subsection 5.5.2 we quantify the effect of unhedgeable investment risk on the hedge potential from including death benefits ($d_g \neq 0$).

5.5.1 Interaction effects in annuity portfolios

In this section we consider portfolios of single life and survivor annuities, and quantify the effect of unhedgeable investment risk on: (i) the required buffer percentage for a given liability mix, and, (ii), the hedge potential that arises from the liability mix. Without death benefits, it follows from (5.14) that the effect of liability mix is fully characterized by the gender mix γ, and by the ratios w_m and w_f of insured rights for survivor annuities over insured rights for single life annuities for males and females, respectively.

Figure 5.5.1 displays the minimum required buffer percentage c as a function of gender mix and product mix in portfolios of single life and survivor annuities. To

[5] Straightforward algebra shows that the aggregate liability payment in year τ in (5.2) is given by (5.14) multiplied by $\sum_{i\in\mathcal{I}}\overline{\delta}_{i,sl}$, the total insured rights for single life annuities. It follows immediately from Proposition 5.3.2 that the minimum required buffer percentage c is unaffected when all liability payments are divided by $\sum_{i\in\mathcal{I}}\overline{\delta}_{i,sl} > 0$.

limit the number of parameters, we consider the case where the product mix is equal for both genders, i.e., $w_m = w_f = w$.

The left panels in Figure 5.5.1 display the minimum required buffer percentage c as a function of gender mix (i.e., γ), for three different product mixes:

- top panel: portfolios with only single life annuities, i.e., with $w = 0$;
- middle panel: portfolios with both single life and survivor annuities where the insured right for survivor annuities is 35% of the insured right for single life annuities, i.e., with $w = 0.35$,
- bottom panel: portfolios with both single life and survivor annuities where the insured right for survivor annuities is 70% of the insured right for single life annuities, i.e., with $w = 0.7$.

The right panels display the minimal required buffer percentage c as a function of product mix (i.e., w), for three different gender mixes:

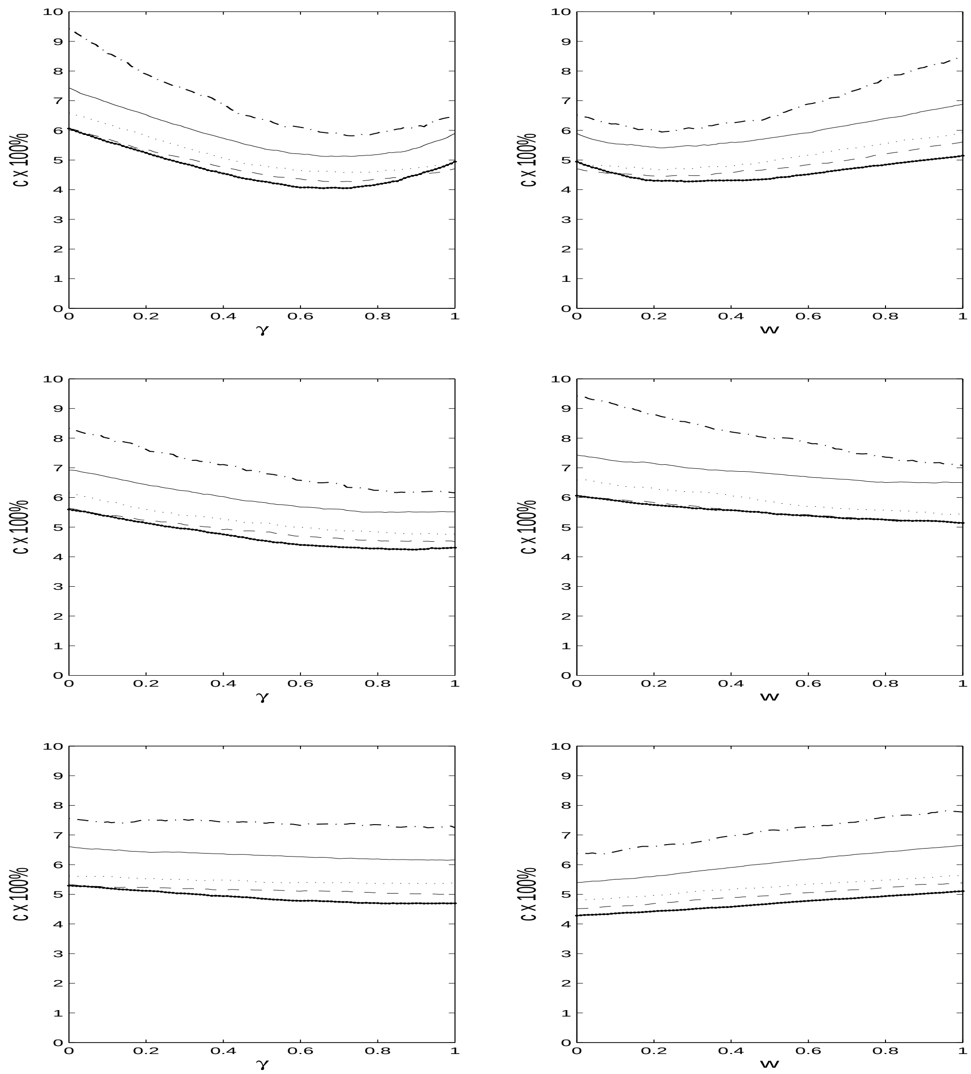
- top panel: portfolios with only male insureds, i.e., with $\gamma = 1$;
- middle panel: portfolios with only female insureds, i.e., with $\gamma = 0$;
- bottom panel: portfolios with 50% male insured rights and 50% female insured rights, i.e., with $\gamma = 0.5$.

In each case we consider four different asset mixes for the buffer portfolio: 100% equity (dashed-dotted lines), 67% equity and 33% one-year zero-coupon bonds (dotted lines), 33% equity and 67% one-year zero-coupon bonds (dashed lines), and 100% one-year zero-coupon bonds (thin solid lines). The bold solid lines lines correspond to the benchmark liability-only case with a constant and deterministic return of $r = 0.04$.

The figure shows that there are important interactions between longevity risk and investment risk. First, the effect of unhedgeable financial risk depends strongly on the liability mix. Second, the effect of liability mix depends nontrivially on the asset mix. Specifically, we observe the following.

Liability mix effects (i.e., effects of gender mix and product mix). For any given asset mix, both gender mix and product mix can significantly affect the required buffer percentage, because different types of liabilities have different sensitivities to changes in mortality rates. Specifically:

Figure 5.5.1: Reserve requirements in portfolios of single life and survivor annuities.



The left panels display the reserve requirements as a function of γ (gender mix). The upper panel represents a fund with only single life annuities ($w = 0$), the middle panel one with single life annuities and 35% survivor annuities ($w = 0.35$), and the bottom panel one with single life annuities and 70% survivor annuities ($w = 0.7$). The right panels display the reserve requirements as a function of w (product mix). The upper panel represents a fund with only males ($\gamma = 1$), the middle one a fund with only females ($\gamma = 0$), and the bottom one a fund with 50% male rights and 50% female rights ($\gamma = 0.5$). The curves correspond to different compositions of the *buffer portfolio*: thin solid curves 100% one-year zero-coupon bonds, dashed curves 67% one-year zero-coupon bonds and 33% equity, dotted curves 33% one-year zero-coupon bonds and 67% equity, dashed-dotted curves 100% equity. The bold solid curves correspond to the liability-only approach.

- For each product mix w, portfolios with exclusively male liabilities ($\gamma = 1$) require lower buffer percentages than portfolios with exclusively female liabilities ($\gamma = 0$). However, in portfolios with only single life annuities (i.e., $w = 0$, top panel), risk is minimized with a mixture of female and male liabilities. This occurs because male and female liabilities are imperfectly correlated, so that there is some diversification effect from combining these liabilities.[6] Including survivor annuities (middle and lower panels) increases the correlation between male and female liabilities and thus reduces the diversification effect. As a consequence, mixing male and female liabilities does not yield significant risk reduction in these cases.

- Combining single life with survivor annuities (right panels) may either increase or decrease the required buffer percentage. This occurs because there are two opposite effects. On the one hand survivor annuities can reduce required buffers because survivor annuity payments are negatively correlated with single life annuity payments.[7] On the other hand, survivor annuity payments are more affected by the uncertainty in future survival probabilities because they have a longer duration (see Table 1). For portfolios with predominantly female rights (middle panel), the former effect dominates; for portfolios with half male and half female rights (bottom panel), the latter effect dominates.

- *Accurate quantification of liability mix effects requires specification of the asset mix.* For example, the middle right panel shows that the potential risk reduction from combining single life annuities with survivor annuities is significantly larger when the buffer portfolio is fully invested in equity than for the other asset mixes that we consider.

[6]The underlying intuition in both cases is as follows. In each case, the random variable of interest can be written as a convex combination $L = \alpha L_1 + (1-\alpha)L_2$ of two present value variables L_1 and L_2. It holds that

$$\mathrm{Var}\{L\} = \alpha^2 \cdot \mathrm{Var}\{L_1\} + (1-\alpha)^2 \cdot \mathrm{Var}\{L_2\} + 2\alpha(1-\alpha) \cdot \mathrm{Cov}\{L_1, L_2\}.$$

Thus, the variance is minimized with an unbalanced portfolio that puts all weight on the liability with the lowest variance if $\mathrm{Cov}\{L_1, L_2\} > \min\{\mathrm{Var}\{L_1\}, \mathrm{Var}\{L_2\}\}$, but the variance is minimized at an internal $\alpha \in (0,1)$ if $\mathrm{Cov}\{L_1, L_2\} < \min\{\mathrm{Var}\{L_1\}, \mathrm{Var}\{L_1\}\}$. Thus, shifting more weight to the higher risk liability is beneficial if the covariance is sufficiently low.

[7]This occurs for two reasons. First, an increase in life expectancy of the insured delays the onset of payments of the survivor annuity, so that they are more heavily discounted. Second, the difference between male and female life expectancies decreases, so that the duration of survivor annuity payments decreases.

Impact of unhedgeable investment risk. We observe two effects:

- For every liability mix, the required buffer percentage is significantly affected by *unhedgeable* investment risk. An increase in equity leads to a higher expected return, but it also yields a higher probability that the realized return is lower than expected. The impact of unhedgeable financial risk is minimized when 1/3 of the buffer is invested in equity.

- *Accurate quantification of the effect of unhedgeable financial risk requires specification of the liability mix.* For example, for portfolios with predominantly female rights, unhedgeable investment risk affects the required buffer more strongly when the fraction of survivor annuity rights is high. The opposite holds for portfolios with half male and half female rights. Although we are not interested in the best equity portfolio, we do observe that the reserve requirement is lower if the insurer invests some of his buffer portfolio in equities. This is due to the use of a particular quantile of L in the determination of the reserve requirements and the equity risk premium. Hence, the investment strategy which reduces the ruin probability will depend on the quantile used in the ruin probability.

These results suggest that separately quantifying investment risk and longevity risk, as is proposed by the Dutch regulator, likely leads to inaccurate quantifications of the impact of longevity risk. Second, ignoring the impact of unhedgeable financial risk may lead to inaccurate quantification of the risk reduction that arises from combining different types of longevity-linked liabilities.

5.5.2 Natural hedge potential of death benefits

In this subsection we investigate the effect of unhedgeable investment risk on the natural hedge potential from death benefits in portfolios of life annuities. To do so, we determine the minimum required buffer percentage c as a function of both asset and liability mix. We then compare the results to the benchmark case considered in the existing literature (for example, Wang et al. 2010, and Tsai et al. 2010), where: (i) longevity risk is quantified with a liability-only approach (i.e., ignoring unhedgeable financial risk), and, (ii) longevity-linked liabilities other than single life annuities and death benefits (such as, for example, survivor annuities) are ignored.

The following proposition shows that in the benchmark case longevity risk in single life annuities can be fully hedged by death benefits.

Proposition 5.5.1 *Let $r_s^{be,(\tau)} = r_s^{bu} = r$, for $s = 0, \cdots, T$, and $\tau = 1, \cdots, T$, and for some (non-random) $r > 0$. Then, for portfolios of single life annuities and death benefits with*

$$d_m = d_f = \frac{1+r}{r}, \tag{5.15}$$

it holds that the terminal asset value A_T is unaffected by longevity risk, and is nonnegative for any $c \geqslant 0$.

Proof. It follows from the proof of Proposition 5.3.2 that $A_T = [(1+c) \cdot BEL - L] \cdot (1+r)^T$, with $L = \sum_{\tau=1}^{T} \frac{\widetilde{L}_\tau}{(1+r)^\tau}$. Moreover, it follows from (5.14), (5.15), and the fact that the portfolio does not contain survivor annuities (i.e., $w_m = w_f = 0$) that:

$$\widetilde{L}_\tau = (1-\gamma)\left[\widetilde{L}_{sl,\tau}(65,m) + \delta \widetilde{L}_{db,\tau}(65,m)\right] + \gamma \cdot \left[\widetilde{L}_{sl,\tau}(65,f) + \delta \widetilde{L}_{db,\tau}(65,f)\right],$$

where $\delta = \frac{1+r}{r}$. Therefore,

$$L = (1-\gamma) \cdot \overline{L}(m) + \gamma \cdot \overline{L}(f),$$

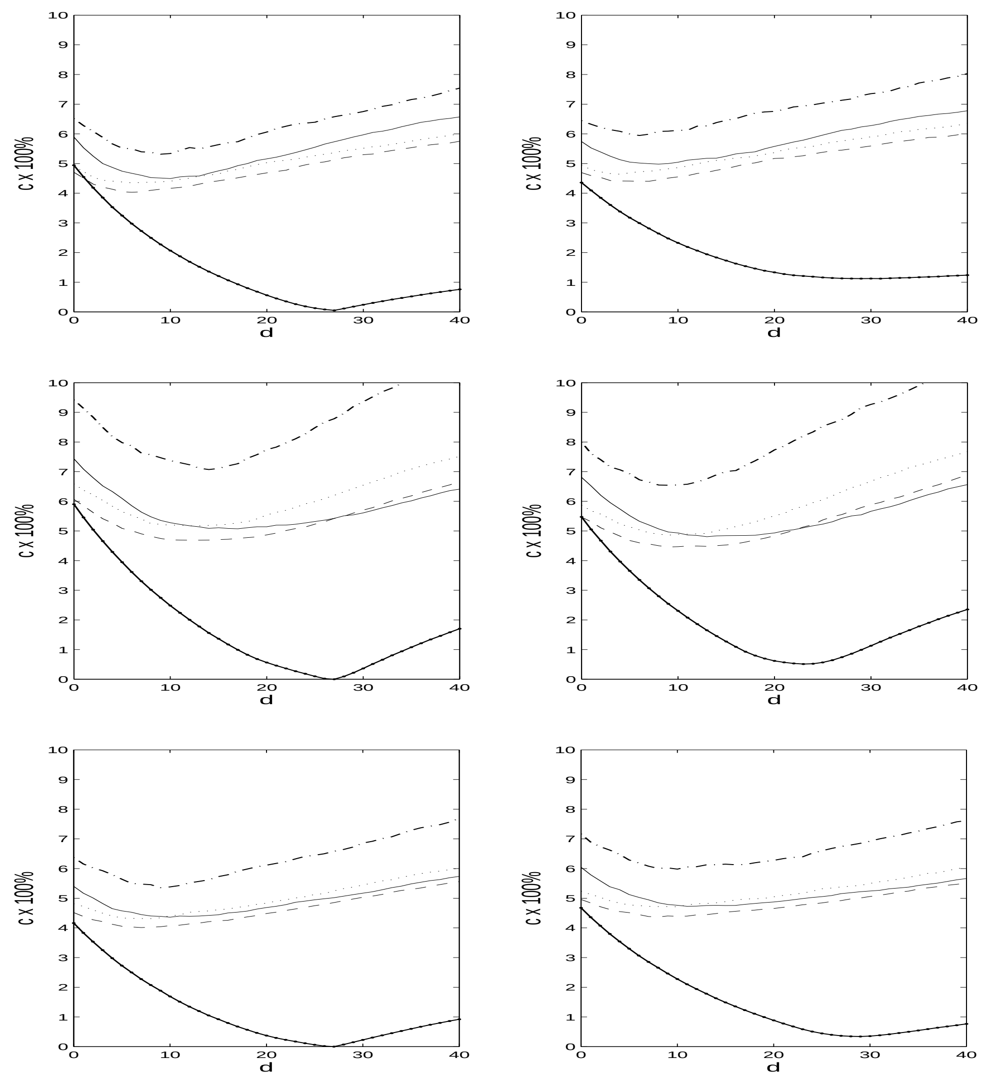
where for $g \in \{m, f\}$, it holds that:

$$\begin{aligned}
\overline{L}(g) &:= \sum_{\tau=1}^{T} \left[\frac{{}_\tau p_{65}^{(g)} + \delta \cdot \left({}_{\tau-1}p_{65}^{(g)} - {}_\tau p_{65}^{(g)} \right)}{(1+r)^\tau} \right] \\
&= \sum_{\tau=1}^{T-1} \left(\frac{\left(1 - \delta + \frac{\delta}{1+r}\right)}{(1+r)^\tau} \right) {}_\tau p_{65}^{(g)} - (1-\delta) \cdot \frac{{}_T p_{65}^{(g)}}{(1+r)^T} + \delta \cdot {}_0 p_{65}^{(g)} \\
&= \delta.
\end{aligned}$$

The last equality follows from $\delta = \frac{1+r}{r}$, ${}_0p_x^{(g)} = 1$, and ${}_Tp_x^{(g)} = 0$. Therefore, $BEL = L = \delta$, and the terminal asset value is given by $A_T = c \cdot \delta \cdot (1+r)^T$, which is deterministic and nonnegative for any $c \geqslant 0$. ■

Proposition 5.5.1 shows that in the benchmark liability-only case that is typically examined in the literature, longevity risk in single life annuities can be fully hedged with death benefits. In the remainder of this section we show that unhedgeable

Figure 5.5.2: Reserve requirements in portfolios of single life and survivor annuities and death benefits



The graphs present reserve requirements as a function of d (ratio of death benefits) in portfolios of life insurance products. The left panels are portfolios with only single life annuities, the right panels are portfolios with single life annuities and survivor annuities ($w = 0.5$). The upper panels correspond to a fund with only males ($\gamma = 1$), the middle panels correspond to a fund with only females ($\gamma = 0$), and the lower panels correspond to a fund with 50% males and 50% females ($\gamma = 0.5$). The curves correspond to different compositions of the *buffer portfolio*: thin solid curves 100% one-years zero-coupon bond, dashed curves 67% one-year zero-coupon bonds and 33% equity, dotted curves 33% one-year zero-coupon bonds and 67% equity, dashed-dotted curves 100% equity. The bold solid curves correspond to the liability-only approach.

investment risk can significantly reduce the hedge potential from death benefits in portfolios of life annuities.

Figure 5.5.2 displays the effect of death benefits on the required buffer percentage c for portfolios of single life and survivor annuities, and for given investment strategies. It considers a case where product mix is identical for both genders, i.e., $w = w_m = w_f$ and $d = d_m = d_f$. The left panels in Figure 5.5.2 display the minimum required buffer percentage c as a function of d, the ratio of the insured rights for death benefits over single life annuities, in portfolios with only single life annuities, i.e., with $w = 0$. The right panels display the minimum required buffer as a function of d, for portfolios of single life annuities and survivor annuities with $w = 0.5$. The top panel corresponds to males (i.e., $\gamma = 1$), the middle panel to females (i.e., $\gamma = 0$), and the bottom panel to portfolios with 50% male rights and 50% female rights ($\gamma = 0.5$). In each case we consider four different investment strategies for the buffer portfolio: 100% equity (dashed-dotted lines), 67% equity and 33% one-year zero-coupon bonds (dotted lines), 33% equity and 67% one-year zero-coupon bonds (dashed lines), and 100% one-year zero-coupon bonds (thin solid lines). The bold solid lines correspond to the benchmark liability-only case with a constant and deterministic return of $r = 0.04$.

In line with results reported in, for example, Wang et al. (2010) and Tsai et al. (2010), we find that death benefits can significantly reduce the required buffer percentages in portfolios of life annuities. However, we find that the risk reduction can be significantly affected by unhedgeable investment risk. Specifically,

- *Ignoring the effect of unhedgeable financial risk leads to significant overestimation of the hedge potential.* Whereas with a liability-only approach to quantify longevity risk (bold solid lines)), the minimum required buffer percentage under the optimal hedge is zero (see Proposition 5.5.1), it varies from around 4% to more than 9%, depending on the asset mix when we take into account the effect of unhedgeable financial risk.

- *Accurate quantification of the hedge potential requires specification of both the existing liability mix and the asset mix.* While the hedge potential from death benefits is generally different for female liabilities (middle row) and for male liabilities (upper row), the difference is much more significant for the risky investment strategy (100% stocks) than for the other strategies that we consider. Also, comparing the left and right panels shows that, depending on the investment strategy, the hedge potential from death benefits may, but need not, decrease significantly when the portfolio also contains survivor annuities.

5.6 Hedge effects of survivor swaps

In this section we investigate the hedge potential from investing in survivor swaps. Dowd, Blake, Cairns, and Dawson (2006) discuss the mechanism and use of survivor swaps as an instruments for managing, hedging, and trading mortality-dependent risks. A survivor swap can be defined as a swap involving at least one future (stochastic) mortality-dependent payment. Given this definition, the most basic case of a survivor swap is an exchange of a single fixed payment for a single mortality-dependent payment. More precisely, let ref denote a reference population. Then, at time $t = 0$, party A agrees with party B that A pays to B at time $\tau > 0$ the amount $K(\tau, ref)$ known at time 0, and B pays to A at the amount $S(\tau, ref)$ which depends on realized mortality until date τ in the reference population, and is thus currently stochastic. The payments made in this agreement are that party B pays A the amount $S(\tau, ref) - K(\tau, ref)$, if $K(\tau, ref) < S(\tau, ref)$, and party A pays B the amount $K(\tau, ref) - S(\tau, ref)$, if $K(\tau, ref) > S(\tau, ref)$. Hence, the payment from party B to party A equals:

$$SS(\tau, ref) = S(\tau, ref) - K(\tau, ref), \tag{5.16}$$

where $S(\tau, ref)$ is the random mortality-dependent payment and $K(\tau, ref)$ is the fixed payment.

The survivor swaps we consider in this chapter is one where the floating leg $S(\tau, ref)$ is the realized survival rate for the 65-year old cohort in the underlying reference population, i.e., $S(\tau, ref) = {}_\tau p_{65}^{(ref)}$. Typically, the fixed leg $K(\tau, ref)$ is determined such that there is no cash transfer at the time of the issue. However, there is currently no publicly traded market in longevity-linked products and hence we do not observe the market price of longevity risk.[8] To avoid making assumptions regarding the price of the swap, we set $K(\tau, ref)$ equal to the current expected value of $S(\tau, ref)$. Then, the payment in period τ of the survivor swap is given by:

$$SS(\tau, ref) = {}_\tau p_{65}^{(ref)} - \mathbb{E}\left[{}_\tau p_{65}^{(ref)} \right], \tag{5.17}$$

[8]For an excellent discussion on issues related to pricing of longevity-linked assets or liabilities, see Bauer, Boerger, and Russ (2010).

and there is a cash transfer at the time of issue which equals the (over the counter) price of the survivor swap. We consider a vanilla survivor swap $VSS(ref)$ that consists of a portfolio of survivor swaps with maturities $\tau = 1, \cdots, T$.

It now remains to specify a reference population. A natural reference group from the point of view of the insurer (party B) is the population of the insurer. However, the insurer may then have more information about the population than the seller (party A) of the survivor swap. Since the insurer may have this private information, buying a survivor swap can be interpreted as a signal that the reference group has low mortality probability, and hence the price of the survivor swaps would be high, see Biffis and Blake (2010). Another problem with the natural reference group from the point of view of the insurer is the tradeability of the survivor swaps; when every life insurer has a different reference group, many different survivor swaps are needed. This would lead to much higher transaction costs for the seller of the survivor swap, since he has to put extra efforts in estimating the size of longevity risk in the survivor swaps (Blake, Cairns, Dowd, and McMinn, 2006). In order to eliminate the private information problem and to increase the tradeability, the whole population of a country is often chosen as reference group, since the information on this reference group is the same for the issuer and buyer of the swap. An example is the first longevity bond[9] issued by European Investment Bank/Bank National de Paris announced in November 2004, which had as reference population the English and Welsh males at age 65 in 2003.

In this section we investigate the effect on solvency capital requirement of vanilla survivor swaps with reference population the Dutch aged 65 in 2006. We use two different vanilla survivor swaps, one with reference group the whole male population aged 65 (i.e., $ref = m$), and another with reference group the whole female population aged 65 (i.e., where $ref = f$). Let s_m (s_f) be the number of vanilla survivor swaps with reference population males (females). Then, the liability payment in year τ, net of payoff from longevity swaps, is given by:

$$\widetilde{L}_\tau = \widetilde{L}_\tau - s_m \cdot SS(\tau, m) - s_f \cdot SS(\tau, f). \tag{5.18}$$

Let $V_{VSS}(s_m, s_f)$ denote the date-0 (over the counter) price of the vanilla survivor

[9] The longevity bond was issued by the EIB and managed by BNP Paribas. The face value was £540 million, and was primarily intended for purchase by U.K. pension funds. The survivor swap involved yearly coupon payments that were tied to an initial annuity payment of £50 million indexed to the survivor rates of English and Welsh males aged 65 years in 2003. The longevity bond was withdrawn prior to issue (Mitchell, Piggott, Sherris, and Yow, 2006).

swap. Then, it follows from Proposition 5.3.2 and (5.6) that the minimal required initial asset value in order to limit the probability of ruin to ε is given by:

$$A_0 = BEL + \overline{c}(s_m, s_f) \cdot BEL + V_{VSS}\left(s_m, s_f\right),$$

where

$$\overline{c}(s_m, s_f) = \frac{Q_{1-\varepsilon}(L(s_m, s_f))}{BEL} - 1,$$

with

$$L(s_m, s_f) = BEL + \sum_{\tau=1}^{T} \frac{\widetilde{L}_\tau - \mathbb{E}\left[\widetilde{L}_\tau\right] - s_m \cdot SS(\tau, m) - s_f \cdot SS(\tau, f)}{\prod_{s=1}^{\tau}\left(1 + r_s^{(bu)}\right)}.$$

Note that $\overline{c}(s_m, s_f) \cdot BEL$ now represents the required buffer in excess of the best estimate of the liabilities *and* the price of the vanilla survivor swap. Note also that a change in the portfolio of swaps not only affects the required buffer, but also the price of the portfolio, $V_{VSS}\left(s_m, s_f\right)$. Because we choose not to make assumptions regarding the price of the survivor swaps, we cannot determine the "optimal" fraction of survivor swaps, i.e., the fraction that minimizes the required asset value A_0. However, for any given portfolio of survivor swaps (s_m, s_f), we can determine the relative attractiveness of the vanilla survivor swaps for different liability mixes and asset mixes. Moreover, for any given asset mix, we can determine the maximum price of the portfolio of survivor swaps under which a lower asset value, i.e., A_0, is sufficient to cover all future liabilities with probability at least $1-\varepsilon$ with survivor swaps than without survivor swaps. This maximum price is given by:

$$V_{VSS}^{\max}\left(s_m, s_f\right) = \left[\overline{c}(0,0) - \overline{c}(s_m, s_f)\right] \cdot BEL. \tag{5.19}$$

In Subsection 5.6.1 we investigate how the hedge effect of survivor swaps depends on the liability and asset mix in a benchmark case without basis risk, i.e., in a setting in which the survival rates of the insured population are identical to those of the reference population. In Subsection 5.6.2 we investigate how these effects are affected by basis risk that arises from differences in the mortality experience in the reference group of the survivor swap and the population of the insurer. In order to focus on the effect of unhedgeable financial risk on the reduction in longevity risk, we consider the investment strategies defined in Section 5.5.

5.6.1 Vanilla survivor swaps and product mix

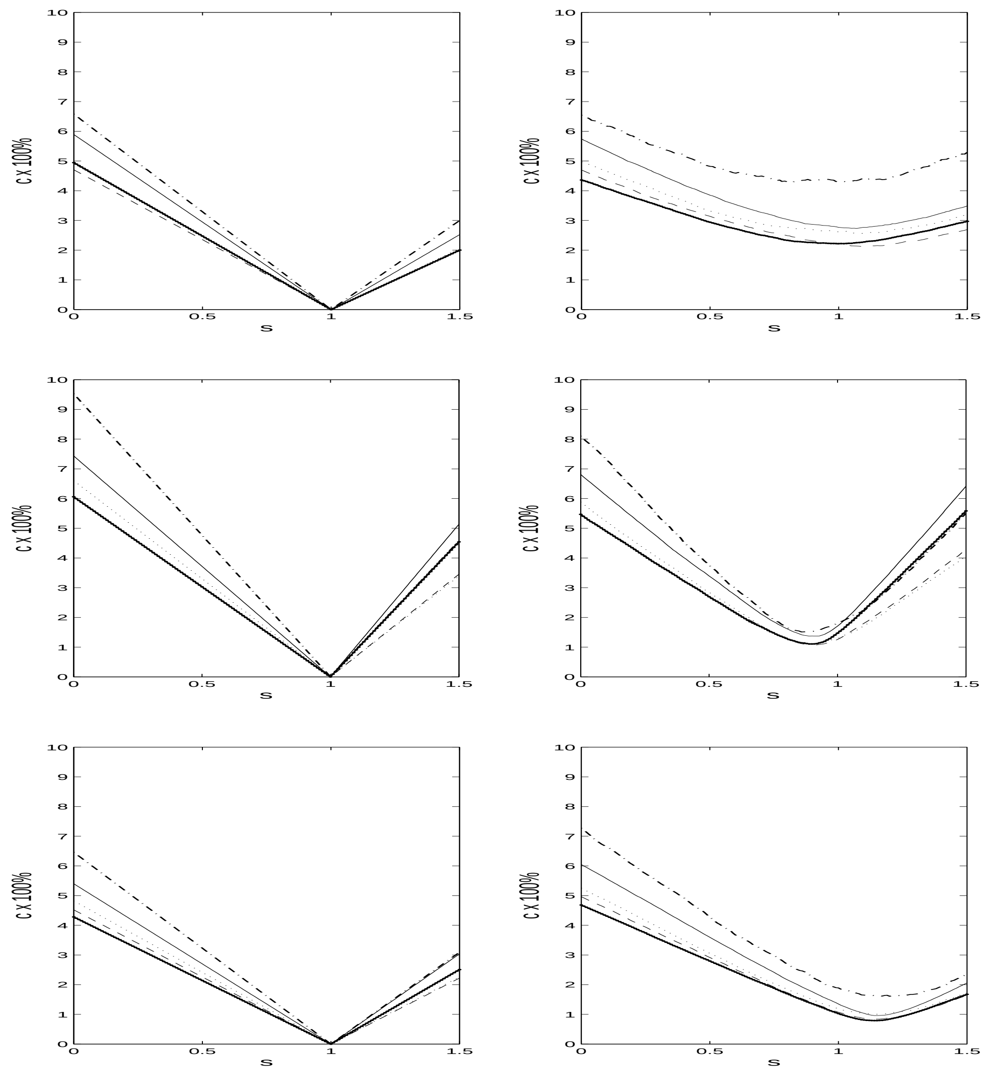
We now investigate the potential hedge effects of survivor swaps for portfolios of life insurance products with different product and gender mixes, and different investment strategies. We also determine the maximum price under which investing in survivor swaps leads to lower capital requirements in each case. In order to reduce the number of parameters, we let $s_m = \gamma \cdot s$, and $s_f = (1-\gamma) \cdot s$. It then follows immediately from (5.1), (5.14), and (5.17), and from the fact that there is no basis risk, that longevity risk in a fraction s of the single life annuity rights for both males and females is fully hedged.[10]

Figures 5.6.1 and 5.6.2 display the minimum required buffer, and the maximum price as defined in (5.19), respectively, as a function of s for different asset and liability mixes, i.e., in portfolios of single life annuities (left panels), and in portfolios of single life and survivor annuities with $w = 0.5$ (right panels), for males (top panel), females (middle panel), and $\gamma = 0.5$ (bottom panel). In each case we consider four different investment strategies for the buffer portfolio: 100% equity (dashed-dotted lines), 67% equity and 33% one-year zero-coupon bonds (dotted lines), 33% equity and 67% one-year zero-coupon bonds (dashed lines), and 100% one-year zero-coupon bonds (thin solid lines). The bold solid lines correspond to the benchmark liability-only case with a constant and deterministic return of $r = 0.04$.

From Figure 5.6.1 we observe that survivor swaps can lead to significant reductions in the required solvency buffer. However, the effect depends strongly on both liability mix and asset mix. Because there is no basis risk, longevity risk in portfolios with only single life annuities (left panels) can be fully eliminated by survivor swaps (with $s = 1$). For portfolios with also survivor annuities, the maximal risk reduction is attained by buying either strictly more or strictly less survivor swaps than the face value of the single life annuities, i.e., with $s < 1$ or $s > 1$. This occurs because survivor annuities to some extent can provide a natural hedge for single life annuities, but on the other hand are also affected more strongly by longevity risk because they have longer duration. The first effect dominates for portfolios with only female insureds, whereas the second effect dominates for portfolios with half male and half female insured rights. Comparing the top left and right panels shows

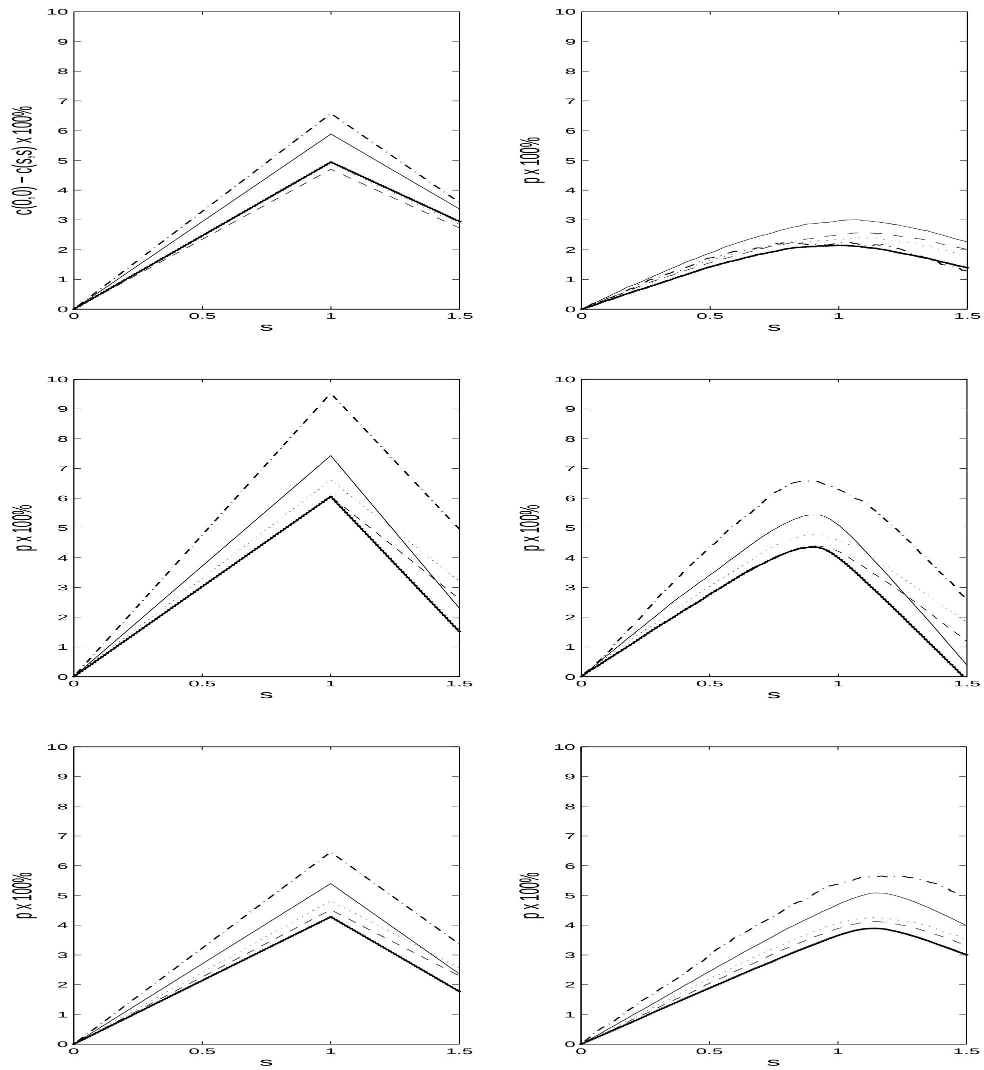
[10] It follows from (5.1), (5.14), and (5.17) that when $s_m = \gamma \cdot s$, and $s_f = (1-\gamma) \cdot s$, a fraction s of the single life annuity payments in year τ, $\gamma \cdot \widetilde{L}_{sl,\tau}(65, m) + (1-\gamma) \cdot \widetilde{L}_{sl,\tau}(65, f)$, is effectively replaced by its expected value, $\gamma \cdot \mathbb{E}\left[\widetilde{L}_{sl,\tau}(65, m)\right] + (1-\gamma) \cdot \mathbb{E}\left[\widetilde{L}_{sl,\tau}(65, f)\right]$.

Figure 5.6.1: Reserve requirements in portfolios of annuities and vanilla survivor swaps without basis risk



The figure displays the reserve requirements, $\overline{c}(s,s)$ as a function of s in a portfolio of life insurance products, for a fund with only single life annuities ($w = 0$), left panels, for a fund with single life and survivor annuities ($w = 0.5$), right panels. The upper row corresponds to a fund with only males ($\gamma = 1$), the middle row to a fund with only females ($\gamma = 0$), and the lower row to a fund with 50% male rights and 50% female rights ($\gamma = 0.5$). The curves correspond to different compositions of the *buffer portfolio*: thin solid curves 100% one-years zero-coupon bond, dashed curves 67% one-year zero-coupon bonds and 33% equity, dotted curves 33% one-year zero-coupon bonds and 67% equity, dashed-dotted curves 100% equity. The bold solid curves correspond to the liability-only approach.

Figure 5.6.2: Maximum price of vanilla survivor swaps without basis risk



The figure displays the maximum price, $p = \overline{c}(0,0) - \overline{c}(s,s)$, as a function of s in a portfolio of life insurance products, for a fund with only single life annuities ($w = 0$), left panels, for a fund with single life and survivor annuities ($w = 0.5$), right panels. The upper row corresponds to a fund with only males ($\gamma = 1$), the middle row to a fund with only females ($\gamma = 0$), and the lower row to a fund with 50% male rights and 50% female rights ($\gamma = 0.5$). The curves correspond to different compositions of the *buffer portfolio*: thin solid curves 100% one-years zero-coupon bond, dashed curves 67% one-year zero-coupon bonds and 33% equity, dotted curves 33% one-year zero-coupon bonds and 67% equity, dashed-dotted curves 100% equity. The bold solid curves correspond to the liability-only approach.

that for male insureds, the hedge potential of survivor swaps reduces dramatically when the portfolio also contains survivor annuities. Comparing the right top and middle panels shows that the hedge potential of survivor swaps in portfolios with both single life annuities and survivor annuities is sufficiently weaker in portfolios with predominantly male insureds.

With regard to the interaction between longevity risk and investment risk, we observe that ignoring the effect of unhedgeable financial risk may lead to both over- or underestimation of the hedge potential of survivor swaps, depending on the investment strategy.

5.6.2 Vanilla survivor swaps with basis risk

In the previous section we showed that vanilla survivor swaps can substantially reduce reserve requirements in portfolios of life insurance products. For portfolios consisting of only single life annuities, they can even eliminate all longevity risk. However, in these calculations we have ignored the impact of *basis risk*, i.e., the mortality rates of the individuals in the reference group for the vanilla survivor swap are assumed to be equal to the mortality rates of the insured population. There is ample empirical evidence, however, that survival rates of insured populations can differ significantly from those of the general population. As discussed above, there are important hurdles to create a liquid market in survivor swaps without basis risk, because that would require fine tuning the survivor swap to the population of the insurer.

Dowd, Cairns, and Blake (2006) investigate the hedge effectiveness of a longevity bond with basis risk that arises because the longevity bond is based on the mortality experience of the cohort of 60-year-old males, and the insured population consists of 65-year-old males. They find that the hedge potential is not significantly affected by this basis risk. In this chapter we quantify the effect of basis risk that arises due to differences in survival probabilities for insured individuals compared to those of the whole population. It is well-documented that, due to adverse selection, survival probabilities of insured individuals are generally different from those of the whole population (see, for example, Brouhns et al. 2002, and Denuit, 2008). Following Brouhns et al. (2002) and Denuit (2008), we will distinguish basis risk in case of group insureds, which is relevant in particular for pension funds, as well as basis risk in case of individual insureds, which is particularly relevant for insurance companies.

	$h = (m, group)$	$h = (f, group)$	$h = (m, individual)$	$h = (f, individual)$
$\alpha^{(h)}$	-0.71755	-0.577829	-1.54351	-1.024695
$\beta^{(h)}$	0.79180	0.843850	0.81849	0.906784

Table 5.6.1: Parameters estimates of the Cox relational model. Source: Denuit (2008).

We use the Cox-type relational model to model mortality rates of the insured population. Specifically, the relationship between the gender-specific mortality rates of insured group h relative to the gender-specific mortality rates for the total (country-wide) population group g, is modeled as (see Brouhns et al. 2002, and Denuit 2008):

$$\log(\mu_{x,t}^{(h)}) = \alpha^{(h)} + \beta^{(h)} \cdot \log(\mu_{x,t}^{(g)}), \tag{5.20}$$

where $\alpha^{(h)}$ denotes the time- and age-independent difference in mortality rates between group g and h, and $\beta^{(h)}$ denotes the speed of the future mortality improvements of the group h relative to the general population with gender g. We use the estimated parameter reported in Denuit (2008), which are given in Table 5.6.1 for group insureds and individual insureds, and for both males and females.[11]

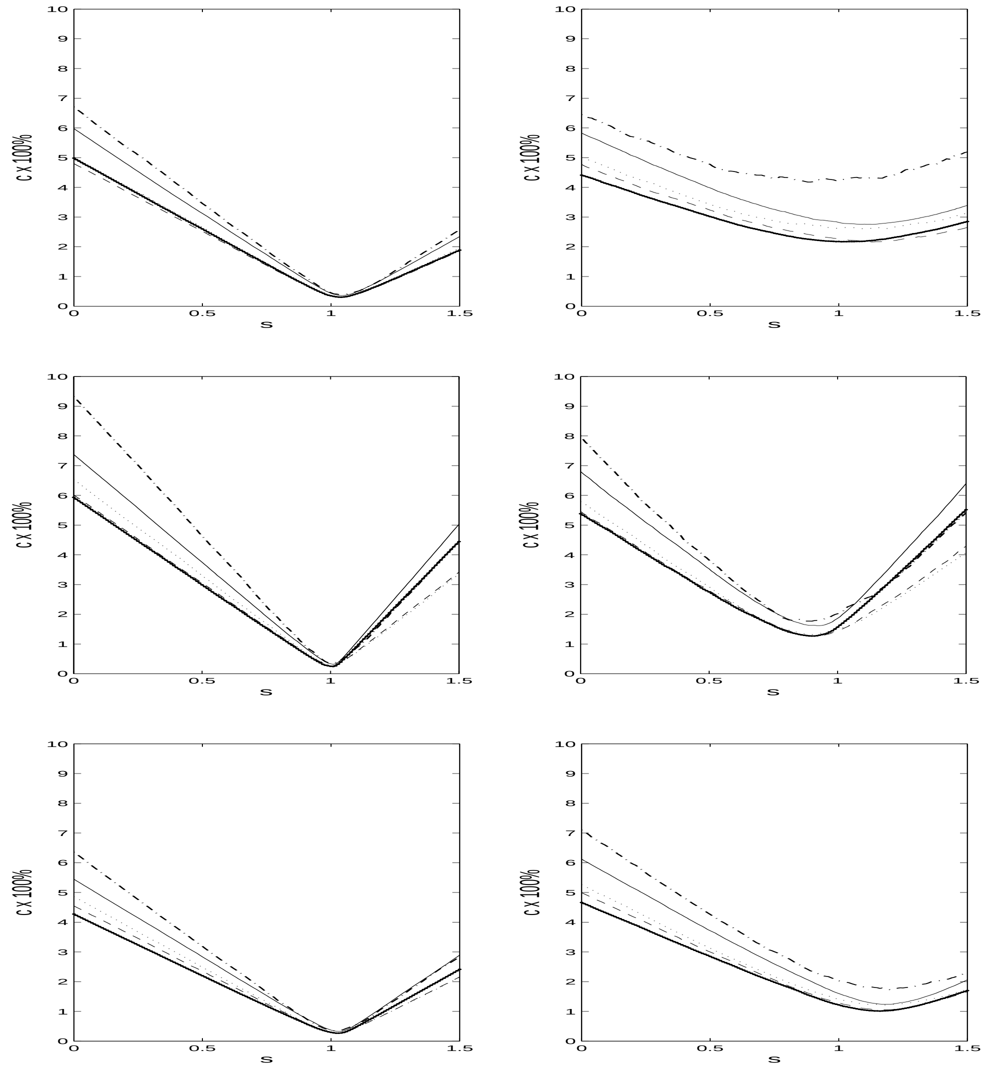
The negative sign of $\alpha^{(h)}$ indicates that the forces of mortality of group and individual insureds are lower than the general population. A larger negative value of $\alpha^{(h)}$ indicates that the difference in the forces of mortality between group h and the general population is larger. The value of $\beta^{(h)}$ smaller than one, in combination with a negative value of $\alpha^{(h)}$, implies that the difference in the forces of mortality between group h and the general population are smaller at old ages than at young ages.

As before, we let $s_m = \gamma \cdot s$, and $s_f = (1 - \gamma) \cdot s$, and we again consider the case where the reference population of the vanilla survivor swap is the general population of males and females, respectively, but we now let mortality rates of the insured persons be given by (5.20).[12]

[11]Notice that $\beta^{(h)} < 1$, which implies that the speed of the future mortality improvements in the insured population is smaller than the corresponding speed for the general population. This occurs because the adverse selection observed in the Belgian individual life market is so strong that the future improvements for the insured population are weaker than for the general population.

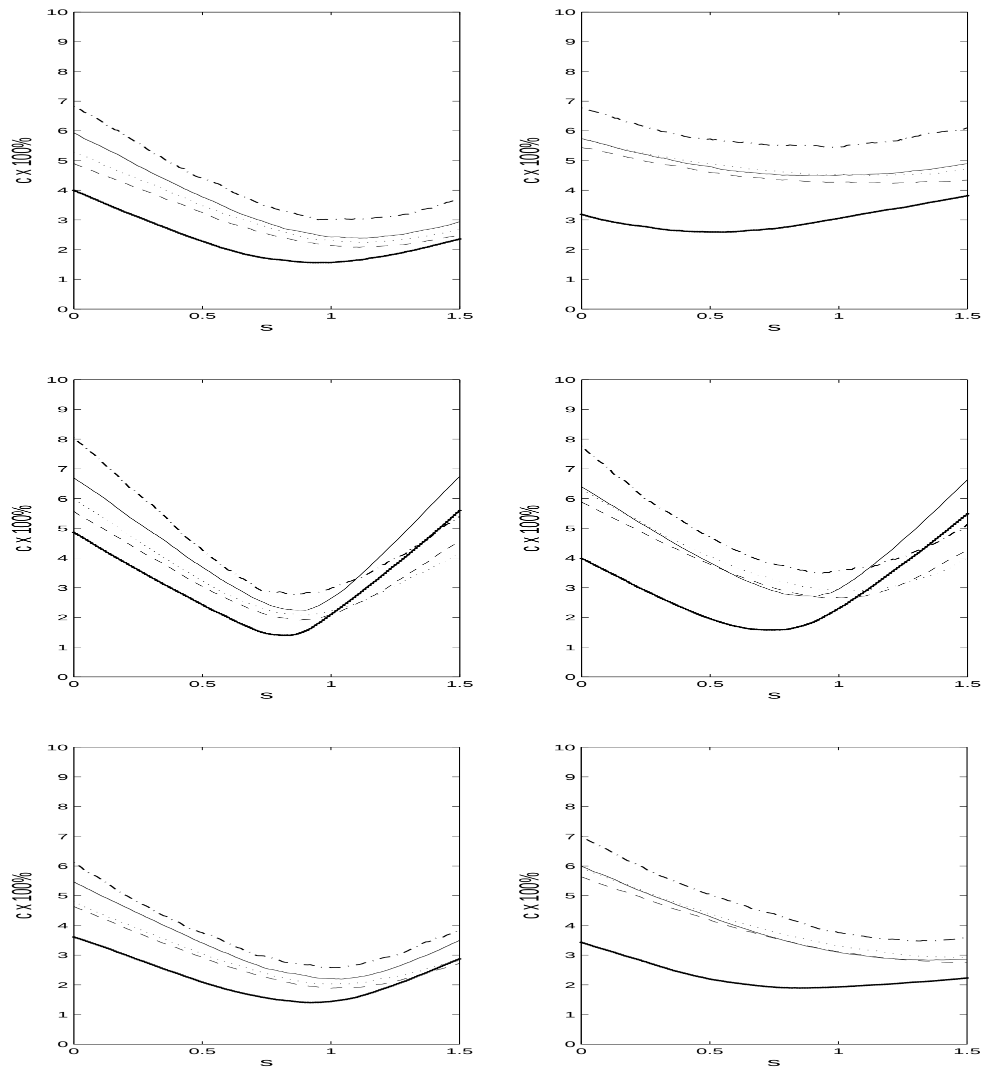
[12]In our model, mortality probabilities of the general population and of the population of the insurer are perfectly correlated. The low hedge effectiveness of the survival swaps is caused by the fact that survival probabilities are non-linear transformations of the logarithm of the forces of mortality. The effect is stronger for portfolios with both single life and survivor annuities because the dependency between males and females.

Figure 5.6.3: Reserve requirements in portfolios of annuities and vanilla survivor swaps with basis risk: group insureds



The figure displays the reserve requirements as a function of s in a portfolio of life insurance products, for a fund with only single life annuities ($w = 0$), left panels, for a fund with single life and survivor annuities ($w = 0.5$), right panels. The upper row corresponds to a fund with only males ($\gamma = 1$), the middle row to a fund with only females ($\gamma = 0$), and the lower row to a fund with 50% male rights and 50% female rights ($\gamma = 0.5$). The curves correspond to different compositions of the *buffer portfolio*: thin solid curves 100% one-years zero-coupon bond, dashed curves 67% one-year zero-coupon bonds and 33% equity, dotted curves 33% one-year zero-coupon bonds and 67% equity, dashed-dotted curves 100% equity. The bold solid curves correspond to the liability-only approach.

Figure 5.6.4: Reserve requirements in portfolios of annuities and vanilla survivor swaps with basis risk: individual insureds



The figure displays the reserve requirements as a function of s in a portfolio of life insurance products, for a fund with only single life annuities ($w = 0$), left panels, for a fund with single life and survivor annuities ($w = 0.5$), right panels. The upper row corresponds to a fund with only males ($\gamma = 1$), the middle row to a fund with only females ($\gamma = 0$), and the lower row to a fund with 50% male rights and 50% female rights ($\gamma = 0.5$). The curves correspond to different compositions of the *buffer portfolio*: thin solid curves 100% one-years zero-coupon bond, dashed curves 67% one-year zero-coupon bonds and 33% equity, dotted curves 33% one-year zero-coupon bonds and 67% equity, dashed-dotted curves 100% equity. The bold solid curves correspond to the liability-only approach.

In Figures 5.6.3 and 5.6.4 we display the minimum required buffer as a function of s, for different asset and liability mixes, i.e., in portfolios of single life annuities (left panels), and in portfolios of single life and survivor annuities with $w = 0.5$ (right panels), for males (top panel), females (middle panel), and $\gamma = 0.5$ (bottom panel). In each case we consider four different investment strategies for the buffer portfolio: 100% equity (dashed-dotted lines), 67% equity and 33% one-year zero-coupon bonds (dotted lines), 33% equity and 67% one-year zero-coupon bonds (dashed lines), and 100% one-year zero-coupon bonds (thin solid lines). The bold solid lines correspond to the benchmark liability-only case with a constant and deterministic return of $r = 0.04$. Figure 5.6.3 corresponds to group insureds, and Figure 5.6.4 corresponds to individual insureds. We assume that if an insured person belongs to group (individual) insureds, the same holds for the insured's partner.[13]

Comparing Figures 5.6.3 and 5.6.4 shows that the hedge effectiveness of survival swaps with basis risk is significantly smaller than without basis risk, especially for portfolios with both single life and survivor annuities.

5.7 Conclusions

This chapter quantifies the effect of longevity risk of portfolios of life insurance products, taking into account that longevity risk induces unhedgeable financial risk. We find that unhedgeable financial risk induces non-trivial interactions between asset mix and liability mix. These interactions affect the impact of longevity risk for any given type of liability, as well as the potential effects of combining different types of liabilities and/or investing in longevity-linked assets.

Our results suggest that analyzing the joint effect of liability mix and asset mix on the overall risk is important for two reasons. First, taking into account interactions between financial risk and longevity risk may lead to more accurate solvency measures. Separating investment risk and longevity risk, as is often proposed by regulators, unavoidably leads to inaccurate quantifications of the impact of longevity risk. Second, ignoring the impact of unhedgeable financial risk may lead to inaccurate quantification of the risk reduction that arises from combining different types of longevity-linked assets and liabilities. Specifically, insurers may be able to reduce their sensitivity to longevity risk by redistributing their risk. Our results indicate

[13]Typically, the mortality probabilities of spouses are similar, due to, for instance, the living conditions.

that the extent to which insurers may benefit from such mutual reinsurance depends not only on their liability portfolios, but also on their investment strategies. Finally, our results indicate that the hedge potential from investing in longevity-linked asset such as survivor swaps depends nontrivially on both the asset mix and the liability mix.

The risk measure in this chapter is based on the ruin probability approach. As discussed in Chapter 2, the existing literature uses different approaches to quantify longevity risk. These include the discounted present value of liabilities approach and the funding ratio approach. The advantage of the ruin probability approach is that it takes into account both assets and liabilities, which is not the case in the discounted present value of liabilities approach. Another advantage of the ruin probability approach is that it does not require the price of the liabilities, including a risk premium for systematic longevity risk, which would be required in the funding ratio approach. However, the ruin probability approach has its limitations. First, it does not take into account any shortfalls before time T. Although in our model, asset value is negative at some date iff it is negative at expiration date, the risk measure does not take into account the probability of technical default when solvency criteria specified by the regulator are violated, e.g., when the value of the assets drops below the value as the liabilities. Second, although we have a dynamic rebalancing investment strategy which depends on the realizations of the survival probabilities and investment returns, it is questionable whether a static asset mix for the buffer portfolio is optimal.

5.A The distribution of the financial returns

In this section we briefly describe the quantification of the financial risk. Financial risk might arise due to investing in (default-free) zero-coupon bonds with different times to maturity or in an equity stock index. The bonds are described by the Vasicek-model, while the stock index is modeled by a Geometric Brownian Motion with time-varying drift. We allow for correlation between the bonds and the stock index.

In case of the Vasicek-model the instantaneous spot rate, r_t, evolves as an Ornstein-Uhlenbeck process with constant coefficients:

$$dr_t = (a - br_t)\, dt + \sigma dZ_t^1, \tag{5.21}$$

where a, b, and σ are model parameters, and Z_t^1 is a standard Brownian Motion. The stock index, S_t, follows a Geometric Brownian Motion with time-varying drift:

$$dS_t = \mu_t S_t dt + \sigma_S S_t dZ_t^2, \quad \mu_t = r_t + \lambda_S \sigma_S, \tag{5.22}$$

where λ_S and σ_S are model parameters, and Z_t^2 is a standard Brownian Motion. The correlation between the standard Brownian Motions Z_t^1 and Z_t^2 is equal to ρ.

Let $P_t^{(n)}$ be the price at time t of a zero-coupon bond with face value of one which matures at time $t+n$, and let $R_t^{(n)}$ be the corresponding yield to maturity $R_t^{(n)}$. Then we have:

$$R_t^{(n)} \equiv \frac{-\log\left(P_t^{(n)}\right)}{n} = \frac{A_t^{(n)}}{n} + \frac{B_t^{(n)}}{n} \cdot r_t, \tag{5.23}$$

with

$$\begin{aligned}
A_t^{(n)} &= \frac{\sigma^2}{4b} \cdot \left(B_t^{(n)}\right)^2 - \left(B_t^{(n)} - n\right) \cdot \left(\frac{(a - \sigma\lambda) \cdot 2b - \sigma^2}{2b^2}\right), \\
B_t^{(n)} &= \frac{1 - \exp\left(-b \cdot n\right)}{b},
\end{aligned}$$

with the additional parameter λ representing the price of risk.

To estimate the parameters of the Ornstein-Uhlenbeck process and the stock index process we discretize the stochastic differential equations (SDE) of equations (5.21) and (5.22). Let Δt be the time step, then we have, with $\alpha = a\Delta t$, $\beta = b\Delta t$, and $\sigma_{\Delta t} = \sigma\sqrt{\Delta t}$:

$$\begin{aligned}
r_{t+\Delta t} - r_t &= \alpha - \beta r_t + \epsilon_{t+\Delta t}, \\
\frac{S_{t+\Delta t} - S_t}{S_t} &= \left(r_t + \lambda_S \sigma_S\right) \Delta t + \epsilon_{t+\Delta t}^S, \\
\begin{pmatrix} \epsilon_{t+\Delta t} \\ \epsilon_{t+\Delta t}^S \end{pmatrix} \mid \mathcal{F}_t &\sim \begin{pmatrix} \sigma_{\Delta t} & 0 \\ 0 & \sigma_{\Delta t}^S \end{pmatrix} \times N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}\right),
\end{aligned}$$

where $\mathcal{F}_t$ denotes the information available at time t, and N stands for a normal distribution. For estimation purposes, we use five implied moment conditions:

$$\begin{array}{lclclcl}
\mathbb{E}\left[\epsilon_{t+\Delta_t}\right] & = & 0, & \quad & \mathbb{E}\left[\epsilon_{t+\Delta_t}^2\right] & = & \sigma_{\Delta t}^2, \\
\mathbb{E}\left[\epsilon_{t+\Delta_t}^S\right] & = & 0, & & \mathbb{E}\left[\left(\epsilon_{t+\Delta_t}^S\right)^2\right] & = & \left(\sigma_{\Delta t}^S\right)^2, \\
\mathbb{E}\left[\epsilon_{t+\Delta_t}\epsilon_{t+\Delta_t}^S\right] & = & \rho\sigma_{\Delta t}\sigma_{\Delta t}^S. & & & &
\end{array} \tag{5.24}$$

Table 5.A.1: Parameter estimates of distribution of the financial returns

Parameter	a	b	σ	λ	λ_S	σ_S	ρ
Estimate	0.0045908	0.10399	0.0042971	-0.81134	0.40832	0.23663	-0.028284
St. dev.	0.0011086	0.026058	0.0005669	0.3307	0.17706	0.017793	0.008286

The table displays the estimates and the standard deviation of the estimates of the model parameters for the distribution of the returns of the assets in the financial market.

In order to estimate the additional parameter λ we assume that the yield on a zero-coupon bond maturing in $n = 10$ years from time t is given by (5.23) plus a mean zero error term $\epsilon_t^{(n)}$:

$$R_t^{(n)} = -\frac{D_t^{(n)}}{n} + \frac{B_t^{(n)} r_t}{n} + \epsilon_t^{(n)}, \ \mathbb{E}\left[\epsilon_t^{(n)}\right] = 0. \tag{5.25}$$

We add to the moment restrictions in (5.24) and (5.25) as extra moment conditions

$$\mathbb{E}\left[\epsilon_{t+\Delta t} r_t\right] = 0, \quad \mathbb{E}\left[\epsilon^R_{t+\Delta t} r_t\right] = 0, \quad \mathbb{E}\left[\left(\epsilon^2_{t+\Delta t} - \sigma^2_{\Delta t}\right) r_t\right] = 0. \tag{5.26}$$

We use daily Dutch financial data obtained from *Datastream* from January 31, 1997 till January 1, 2007. We use three time series, namely the one month interest rate, the interest rate on a 10 years Dutch government bond, and the return on the Dutch stock index "AEX." When estimating the model parameters using the Generalized Method of Moments (GMM) (with optimal weighting matrix) based on the moment restrictions (5.24)–(5.26), we make use of the Newey-West covariance matrix estimator. We experimented with the lag length in this estimator. The reported estimates correspond to lag length equal to ... Table 5.A.1 displays the estimates and the standard deviation of the estimates of the model parameters.

We include two sources of financial risk: process risk and parameter risk. First, using (5.22) and (5.23) and using the GMM-based estimates, there is *process risk* due to the fact that future values of r_t and S_t are risky. Next, these forecasts are based on estimates sensitive to estimation inaccuracy. The corresponding risk is referred to as *parameter risk*. Let θ be the vector of all parameters estimated by GMM. The GMM-estimator $\widehat{\theta}_{GMM}$ satisfies $\sqrt{T}\left(\widehat{\theta}_{GMM} - \theta\right) \xrightarrow{d} N\left(0, V_\theta\right)$. Let $\widehat{V}_\theta$ be a consistent estimator of V_θ. To quantify the financial risk, we simulate 15,000 scenarios as follows. First, we simulate a θ from the $N\left(\widehat{\theta}_{GMM}, \widehat{V}_\theta/T\right)$-distribution, to incorporate parameter risk, and then, given this θ, we simulate the relevant future values of r_t, $R_t^{(n)}$, and S_t, using (5.21)–(5.23), to incorporate process risk.

5.B The distribution of the mortality probabilities

In this section we describe the models used to quantify the systematic longevity risk affecting $p_{x,t}^{(g)}$. Let $\mu_{x,t}^{(g)}$ denote the force of mortality of a person with age x and gender g at time t. We assume that for any integer age x, any gender g, and any time t, it holds that $\mu_{x+u,t}^{(g)} = \mu_{x,t}^{(g)}$, for all $u \in [0,1)$. Then one can verify (see, for example, Pitacco, Denuit, Haberman, and Olivieri, 2009)

$$p_{x,t}^{(g)} = \exp\left(-\mu_{x,t}^{(g)}\right) = \exp\left(-m_{x,t}^{(g)}\right), \tag{5.27}$$

where $m_{x,t}^{(g)}$ is the central death rate. This rate is given by $m_{x,t}^{(g)} = D_{x,t}^{(g)}/E_{x,t}^{(g)}$, with $D_{x,t}^{(g)}$ the observed number of deaths in year t in the cohort with gender g and aged x at the beginning of year t, and with $E_{x,t}^{(g)}$ the corresponding number of person years, the so-called exposure. We use three variants of the Lee and Carter (1992)-model, a P-Spline model, based on Currie et al. (2004), and four variants of the Cairns, Blake, and Dowd (2006) (CBD)-model to quantify the systematic longevity risk. The three variants of the Lee-Carter model are described in Appendix 5.B.1. In Appendix 5.B.2 we describe the P-splines model. In Appendix 5.B.3 we describe the four models for the CBD-model. In Appendix 5.B.4 we then describe our approach of simulating scenarios to generate longevity risk, including model, parameter, and process risk.

5.B.1 Lee-Carter (1992) model

In this section we describe the three variants of the Lee-Carter model, namely the models proposed by Lee and Carter (1992), Brouhns, Denuit, and Vermunt (2002), and Cossette et al. (2007). The model by Lee and Carter (1992) is given by

$$\log\left(m_{x,t}^{(g)}\right) = a_x^{(g)} + b_x^{(g)} k_t^{(g)} + \epsilon_{x,t}^{(g)}, \tag{5.28}$$

where $k_t^{(g)}$ is an index of the level of mortality, $a_x^{(g)}$ is an age-specific constant describing the general pattern of mortality by age, $b_x^{(g)}$ is an age-specific constant describing the relative speed of the change in mortality by age, and where $\epsilon_{x,t}^{(g)}$ represents the measurement error, assumed to satisfy $\epsilon_{x,t}^{(g)} \mid \mathcal{K}_t \sim N\left(0, \sigma_{x,g}^2\right)$, conditional on $\mathcal{K}_t = \left\{k_\tau^{(g)} \mid g \in \{m,f\}, \tau = t, t-1, ...\right\}$. Moreover, we assume that the $\epsilon_{x,t}^{(g)}$ are independent for different x and g, conditional on $\mathcal{K}_t$.

To model the process for $\left(k_t^{(m)}, k_t^{(f)}\right)'$ over time, we use an ARIMA(0,1,1) model (as best fitting ARIMA-model)

$$k_t^{(m)} = k_{t-1}^{(m)} + c^{(m)} + e_t^{(m)} + \theta^{(m)} e_{t-1}^{(m)}, \tag{5.29}$$
$$k_t^{(f)} = k_{t-1}^{(f)} + c^{(f)} + e_t^{(f)} + \theta^{(f)} e_{t-1}^{(f)}, \tag{5.30}$$

where $c^{(g)}$ is the gender g specific drift term which indicates the average annual change of $k_t^{(g)}$, $\theta^{(g)}$ is the gender specific moving average coefficient, and $e_t^{(g)}$ is the gender specific innovation such that

$$\begin{pmatrix} e_t^{(m)} \\ e_t^{(f)} \end{pmatrix} \mid \mathcal{K}_{t-1} \sim \left(\begin{array}{cc} \sigma_m & 0 \\ 0 & \sigma_f \end{array} \right) \times N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & \rho_{mf} \\ \rho_{mf} & 1 \end{pmatrix} \right),$$

where σ_g is the gender-specific standard deviation of the error term $e_t^{(g)}$, and where ρ_{mf} captures the correlation between $e_t^{(m)}$ and $e_t^{(f)}$.

In case of the model by Brouhns, Denuit, and Vermunt (2002), the age and gender specific numbers of deaths are modeled by a Poisson process,

$$D_{x,t}^{(g)} \mid \widetilde{\mathcal{K}}_t \sim \text{Poisson}\left(E_{x,t}^{(g)} e^{a_x^{(g)} + b_x^{(g)} k_t^{(g)}} \right), \tag{5.31}$$

with $\widetilde{\mathcal{K}}_t = \mathcal{K}_t \bigcup \left\{ E_{x,\tau}^{(g)} \mid g \in \{m, f\}, \text{all } x, \tau = t, t-1, ... \right\}$. We assume that the $D_{x,t}^{(g)}$ are independent for different x and g, conditional on $\widetilde{\mathcal{K}}_t$. The process for $\left(k_t^{(m)}, k_t^{(f)}\right)'$ is modeled as in case of the Lee and Carter (1992)-model, i.e., via equations (5.29)–(5.30).

As third model, we consider Cossette et al. (2007). These authors model the age specific numbers of deaths $D_{x,t}^{(g)}$ via the Binomial Gumbel process,

$$D_{x,t}^{(g)} \mid \widetilde{\mathcal{K}}_t \sim \text{Bin}\left(E_{x,t}^{(g)}, 1 - \exp\left(-e^{a_x^{(g)} + b_x^{(g)} k_t^{(g)}} \right) \right), \tag{5.32}$$

where we again assume that the $D_{x,t}^{(g)}$ are independent for different x and g, conditional on $\widetilde{\mathcal{K}}_t$, and where we model the process for $\left(k_t^{(m)}, k_t^{(f)}\right)'$ via equations (5.29)–(5.30).

The model-specific parameters are estimated imposing the required normalizations and using the estimation techniques as described in the corresponding papers. In order to avoid localized age induced anomalies in $\widehat{b}_x^{(g)}$ in the three models, we follow Renshaw and Haberman (2003). These authors proposed to smooth the age

Table 5.B.1: Estimation results for the Lee-Carter models

Model	g	$c^{(g)}$	$\theta^{(g)}$	σ_g	ρ
Lee-Carter	m	-1.854	-0.131	1.612	0.881
	f	-1.576	-0.373	1.779	
Brouhns, Denuit, and Vermunt	m	-1.849	-0.096	1.376	0.897
	f	-1.519	-0.148	1.572	
Cossette et al.	m	-1.854	-0.097	1.386	0.916
	f	-1.529	-0.160	1.594	

Parameter estimates of equations (5.29)–(5.30)). Lee-Carter: Lee and Carter (1992)-model; Brouhns, Denuit, and Vermunt: Brouhns, Denuit, and Vermunt (2002)-model; Cossette et al.: Cossette et al. (2007)-model.

specific estimated parameters $\widehat{b}_x^{(g)}$ using cubic B-splines, with internal knots,

$$\zeta_0^{(g)} + \zeta_1^{(g)} x + \zeta_2^{(g)} x^2 + \zeta_3^{(g)} x^3 + \sum_{j=1}^{r} \zeta_{3+j}^{(g)} (x - x_j)_+^3, \tag{5.33}$$

where $(x - x_j)_+^3 = (x - x_j)^3$, in case $x - x_j > 0$, and zero otherwise. As internal knots we use $x_1 = 9.5$, $x_2 = 20.5$, $x_3 = 50.5$, $x_4 = 60.5$, and $x_r = x_5 = 80.5$. The cubic B-splines are fitted to the (model specific) estimated $\widehat{b}_x^{(g)}$ using the method of least squares.

Age, gender, and time specific numbers of death and exposed to death are obtained from the Human Mortality Database.[14] In our case $x \in \{0, 1, 2, ..., 99, 100^+\}$, with 100^+ the age group of people aged 100 years or more. We use the time period 1977–2006, so that $T = 2006$. This time period minimizes the statistic proposed by Booth et al. (2002) to test the hypothesis that the age components in the original Lee-Carter model are invariant over time. The parameter estimates relevant for the quantification of the systematic longevity risk are plotted in Figure 5.B.1 (the $\widehat{b}_x^{(g)}$) and Table I (the parameter estimates of equations (5.29)–(5.30)).

To forecast the future mortality probabilities, we use (5.27), combined with (5.28), (5.31), or (5.32) (depending on the model), together with (5.29)–(5.30) and (5.33). Let $\widehat{q}_{x,T+s}^{(g)} = 1 - \widehat{p}_{x,T+s}^{(g)}$ be the s-periods ahead model-specific forecasted one-year death probability (starting from the end of the sample $T = 2006$). To avoid a jump-off bias in the forecasts, we correct this forecast using as correction factor $q_{x,T}^{(g)} / \widehat{q}_{x,T}^{(g)}$, with $q_{x,T}^{(g)}$ the observed one-year death probability in year T and $\widehat{q}_{x,T}^{(g)}$ the corresponding model-specific one-year death probability.

[14] See www.mortality.org.

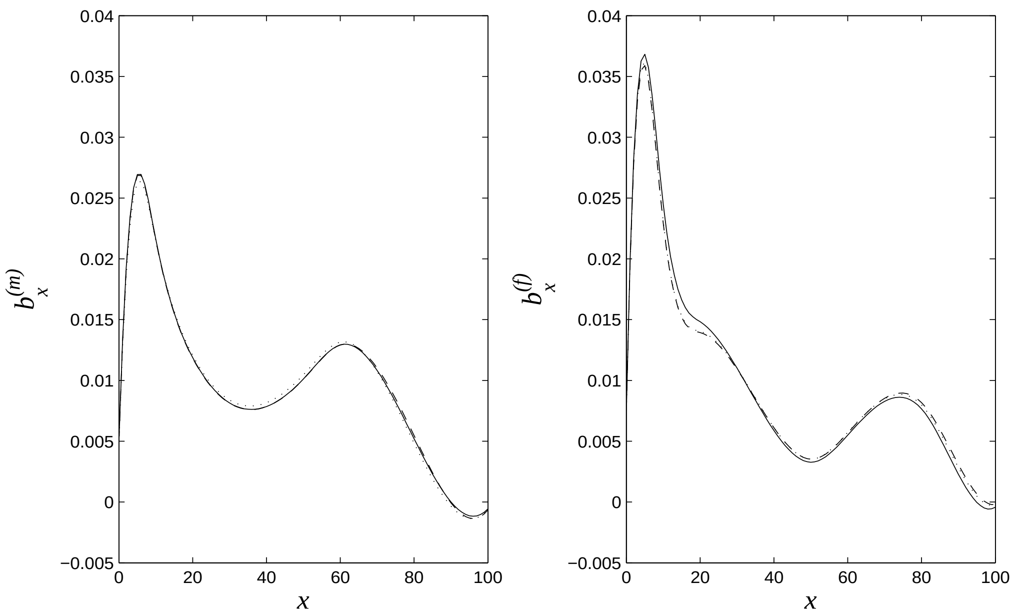


Figure 5.B.1: Estimated $b_x^{(g)}$ after smoothing using cubic B-splines. Left panel: $g = m$; right panel: $g = f$. The solid curve corresponds to the Lee and Carter (1992)-model; the dashed curve corresponds to the Brouhns, Denuit, and Vermunt (2002)-model, and the dotted curve corresponds to the Cossette et al. (2007)-model.

5.B.2 P-Splines

In this section we describe the P-spline model proposed by Currie, Durbin, and Eilers (2004). Let $B_y = B_y(x_y)$, be a $n_y \times c_y$ regression matrix of B-splines based on explanatory variable x_y and let $B_a = B_a(x_a)$, be a $n_a \times c_a$ regression matrix of B-splines based on explanatory variable x_a. The regression matrix for our model is the Kronecker product:

$$B = B_y \otimes B_a.$$

For the general population we assume:

$$\underline{D}^{(m)} + \underline{D}^{(f)} \mid \underline{E}^{(m)} + \underline{E}^{(f)} \sim \text{Poisson}\left(\left(\underline{E}^{(m)} + \underline{E}^{(f)}\right) \exp\left(B\underline{\alpha}^{(p)}\right)\right), \tag{5.34}$$

where the data is arranged in column order, that is $\underline{D}^{(g)} = \text{vec}\left(D^{(g)}\right)$ and $\underline{E}^{(g)} = \text{vec}\left(E^{(g)}\right)$, and the log of a vector is the log applied componentwise. The general trend in the force of mortality of the whole population is given by $B\underline{\alpha}^{(p)}$. For the difference in the forces of mortality between the general population and the gender specific forces of mortality we regress for both $g = m$ and $g = f$:

$$\underline{D}^{(g)} \mid \underline{E}^{(g)} \sim \text{Poisson}\left(\underline{E}^{(g)} \exp\left(B\underline{\widehat{\alpha}}^{(p)} + B\underline{\alpha}^{(g)}\right)\right), \tag{5.35}$$

where $B\underline{\alpha}^{(p)}$ is estimated in the previous step and, thus, assumed to be known in the second step.

To avoid under-smoothing, we use a penalty on α of the form $\alpha' P\alpha$, where the penalty matrix P is given by:

$$P = \lambda_a I_{c_y} \otimes D_a^{'} D_a + \lambda_y D_y^{'} D_y \otimes I_{c_a},$$

with λ_a and λ_y smoothing parameters, I_{c_y} an identity matrix of size c_y, D_a a so-called difference matrix of dimension $(c_a - p_a) \times c_a$ (that takes the column-wise difference of another matrix when post-multiplied), where p_a is the order of the penalty on age, and with I_{c_a} and D_y defined similarly. Given the smoothing parameters λ_a and λ_y, the parameter vector α is estimated by maximizing the log-likelihood based on (5.34) or (5.35) (with $B\underline{\widehat{\alpha}}^{(p)}$ given), corrected for the penalty $\frac{1}{2}\alpha' P\alpha$. The smoothing parameters λ_a and λ_y are set such that they optimize the Bayesian Information Criterion (BIC).

Currie, Durbin, and Eilers (2004) provide an easy way not only to estimate α, but also to calculate forecasts given α. Moreover, these authors provide an approximate normal distribution by which the sampling inaccuracy in the estimate $\widehat{\alpha}$ can be quantified.

The application of the P-spline method requires a large number of settings. Table 5.B.2 presents the settings that we used. As data we used the Dutch mortality data from 1977 till 2006 for the ages 20 till 110.

5.B.3 CBD models

In this section we describe the third class of models, the CBD-models, first introduced in Cairns, Blake, and Dowd (2006). Later several extensions have been proposed, see for example, Cairns et al. (2009). The CBD models fit the one-year mortality probabilities $q_{x,t}^{(g)} = 1 - p_{x,t}^{(g)}$. The general specification of the CBD-model is given by

$$\log\left(\frac{q_{x,t}^{(g)}}{1 - q_{x,t}^{(g)}}\right) = \beta_{1,x}^{(g)}\kappa_{1,t}^{(g)} + \beta_{2,x}^{(g)}\kappa_{2,t}^{(g)} + \beta_{3,x}^{(g)}\kappa_{3,t}^{(g)} + \beta_{4,x}^{(g)}\gamma_{t-x}^{(g)}, \tag{5.36}$$

Table 5.B.2: Parameter settings and Output P-spline model

	General	Males-general	Females-general
$bdeg_a$	3	3	3
$pord_a$	2	2	2
n_a	91	91	91
c_a	21	21	21
λ_a	15	1400	820
$bdeg_a$	3	3	3
$pord_y$	2	2	2
n_y	30	30	30
c_y	8	8	8
λ_y	72	3000	1800

This table displays the parameter settings and output of the P-spline model.

where $\beta_{j,x}^{(g)}$, $j = 1, \cdots, 4$, are possibly age dependent constants, and $\kappa_{j,t}^{(g)}$, $j = 1, 2, 3$, represent time effects, $\gamma_{t-x}^{(g)}$ is a cohort effect, and $\epsilon_{x,t}^{(g)}$ is a residual. We consider the four following possibilities. We define the set C as the set of all cohort years that have been included in the analysis, i.e., $C = \{c = t - x \mid t \in \mathcal{T}, x \in \mathcal{X}\}$, where $\mathcal{T}$ is the sample period and $\mathcal{X}$ is the set of ages considered.

1) $\beta_{1,x}^{(g)} = 1$, $\beta_{2,x}^{(g)} = x - \overline{x}$, $\beta_{3,x}^{(g)} = \beta_{4,x}^{(g)} = 0$ (where $\overline{x}$ is the mean of the ages in $\mathcal{X}$).

2) As 1) but with $\beta_{4,x}^{(g)} = 1$, together with the identification constraints $\sum_{c \in C} \gamma_c^{(g)} = \sum_{c \in C} c \cdot \gamma_c^{(g)} = 0$.

3) As 2) but with $\beta_{3,x}^{(g)} = (x - \overline{x})^2 - \sigma_x^2$ (where σ_x^2 is the variance of the ages in $\mathcal{X}$), together with the extra identification constraint $\sum_{c \in C} c^2 \cdot \gamma_c^{(g)} = 0$.

4) As 2) but with $\beta_{4,x}^{(g)} = C^{(g)} - x$, for some constant parameter $C^{(g)}$, together with the (single) identification constraint $\sum_{c \in C} \gamma_c^{(g)} = 0$.

Version 1) is the original CBD-model, proposed in Cairns, Blake, and Dowd (2006). Let $\kappa_t = \left(\kappa_{1,t}^{(m)}, \kappa_{2,t}^{(m)}, \kappa_{3,t}^{(m)}, \kappa_{1,t}^{(f)}, \kappa_{2,t}^{(f)}, \kappa_{3,t}^{(f)}\right)'$, and $\mathcal{K}_t = \{\kappa_\tau \mid \tau = t, t-1, \cdots\}$. Similar to the model by Brouhns, Denuit, and Vermunt (2002), the age and gender specific numbers of deaths are modeled by a Poisson process,

$$D_{x,t}^{(g)} \mid \widetilde{\mathcal{K}}_t \sim \text{Poisson}\left(m_{x,t}^{(g)} E_{x,t}^{(g)}\right),$$

with $\widetilde{\mathcal{K}}_t = \mathcal{K}_t \bigcup \left\{ E_{x,\tau}^{(g)} \mid g \in \{m, f\}, \text{all } x, \tau = t, t-1, ... \right\}$, together with the assumption that the $D_{x,t}^{(g)}$ are independent for different x and g, conditional on $\widetilde{\mathcal{K}}_t$. Here, $m_{x,t}^{(g)}$ is linked to $q_{x,t}^{(g)}$ via $m_{x,t}^{(g)} = -\log\left(1 - \widehat{q}_{x,t}^{(g)}\right)$, cf. (5.27). The parameters κ_t, for $t \in \mathcal{T}$, γ_c, for $c \in C$, and $C^{(g)}$ are estimated by maximizing the corresponding log likelihood, where we use for $\mathcal{T}$ the sample period from 1977 until 2006 and for the set $\mathcal{X}$ of ages the ages 60 until 100+.

In terms of κ_t, we assume, cf. (5.29)–(5.30),

$$\kappa_t = \kappa_{t-1} + \mu + e_t, \quad e_t \mid \mathcal{K}_{t-1} \sim N(0, V), \tag{5.37}$$

where μ and V represent the mean vector and covariance matrix of $D_t = \kappa_t - \kappa_{t-1}$. Following Cairns, Blake, and Dowd (2006) we assume as non-informative prior distribution for (μ, V) the Jeffreys prior:

$$p(\mu, V) \propto |V|^{-3/2},$$

where $|V|$ is the determinant of the covariance matrix V. The posterior distribution for $(\mu, V|D)$, with $D = (D_1, \cdots, D_T)$, then satisfies

$$\begin{aligned} V^{-1}|D \sim & \text{Wishart}\left(T-1, T^{-1}\widehat{V}^{-1}\right), \\ \mu|V, D \sim & MVN\left(\widehat{\mu}, T^{-1}V\right), \\ \text{where } \widehat{\mu} = & T^{-1}\sum_{t=1}^{n} D_t, \\ \text{and } \widehat{V} = & T^{-1}\sum_{t=1}^{T} (D_t - \widehat{\mu})(D_t - \widehat{\mu})'. \end{aligned}$$

Table 5.B.3 displays the estimates of μ for the different models.

5.B.4 Quantifying Longevity Risk

We include three sources of systematic longevity risk: process risk, parameter risk, and model risk. First, given a specific model and given the corresponding model specific estimates, there is *process risk* due to fact that future values of $\widehat{q}_t^{(g)}$ are still risky. Next, given a specific model, the forecasts of $\widehat{q}_t^{(g)}$ are based on model specific estimates, sensitive to estimation inaccuracy. The corresponding risk is referred to as *parameter risk*. Finally, different models might be used to calculate the forecasts, resulting. Assuming that some prior distribution is used to do the forecast calculations, there is in *model risk*.

Table 5.B.3: Parameter estimates of the CBD-models

	$\mu_1^{(m)} \cdot 10^2$	$\mu_2^{(m)} \cdot 10^4$	$\mu_3^{(m)} \cdot 10^5$	$\mu_1^{(f)} \cdot 10^2$	$\mu_2^{(f)} \cdot 10^4$	$\mu_3^{(f)} \cdot 10^5$
CBD 1	-1.3723	8.3578		-1.1211	1.3925	
CBD 2	-1.3203	3.9099		-1.0141	17.7359	
CBD 3	-1.3708	8.0533	2.1365	-0.87047	1.7736	-5.8667
CBD 4	-3.9336	-1.7694		6.7977	36.4297	

	LogL	# par	BIC
CBD 1	-12042	120	24905
CBD 2	-9344	236	20302
CBD 3	-9220	294	20449
CBD 4	-9431	240	20503

The table displays the estimation of the parameter μ and the log likelihood, number of parameter, and the Bayesian Information Criterion (BIC) for the different CBD-models. For model CBD 4 we have used $C^{(m)} = 74$ and $C^{(f)} = 75$.

To incorporate model risk, we generate 5000 scenarios from each class of models: 5000 scenarios from the Lee-Carter (1992)-type models (1666 scenarios from the Lee-Carter (1992) model, 1666 scenarios from the Brouhns, Denuit, and Vermunt (2002) model, and 1667 scenarios from the Cossete et al. (2007) model); 5000 scenarios from Cairns-Blake-Dowd (2006) models (1250 scenarios from each of the four variants), and 5000 scenarios from the P-Splines model.

To incorporate parameter risk, we simulate in each of the scenarios parameters in a model-specific way. For example, in case of the Lee and Carter (1992) model we simulate $\alpha_x^{(g)}$, $\beta_x^{(g)}$, $\sigma_{x,g}^2$, $c^{(g)}$, $\theta^{(g)}$, σ_g, and ρ_{mf}, using a bootstrap procedure, following Koissi, Shapiro, and Högnäs (2006). A similar approach is used in case of the Brouhns, Denuit, and Vermunt (2002) model and Cossete et al. (2007) model. In case of the P-Splines model we simulate α-s, using the approximate normal distribution of the estimated $\widehat{\alpha}$. In case of the CBD-models we simulate μ and V from the corresponding posterior distribution.

To incorporate process risk, we simulate in case of the Lee-Carter (1992) model, given the simulated parameter values, future values of $k_t^{(g)}$ (by simulating future values of $e_t^{(g)}$) and future values of $\varepsilon_{x,t}^{(g)}$. This results in scenario-specific future values of $q_{x,t}^{(g)}$. In case of the Brouhns, Denuit, and Vermunt (2002) model and Cossete et al. (2007) model we proceed in a similar way. However, in these models we ignore the potential process risk in the error terms $\varepsilon_{x,t}^{(g)}$, which are set equal to zero (in fact, we did not present these error terms in these cases). In case of the P-Splines model the simulated α-s also incorporate process risk. In case of the CBD-models we simulate, given the simulated μ and V, future values of κ_t (by simulating future values of e_t). Similar to the Brouhns, Denuit, and Vermunt (2002) and Cossete et al. (2007) models, we ignore both in the P-spline model and the CBD-models the potential process risk in the error terms $\varepsilon_{x,t}^{(g)}$ (these error terms are also not presented in case of these models).

Chapter 6

Annuity decisions with systematic longevity risk

This chapter is based on Stevens (2010).

6.1 Introduction

Our goal in this chapter is to investigate the optimal annuity decision in a life-cycle model when there is systematic longevity risk. Life expectancy has increased substantially over the past decades, and is expected to increase further in the future. However, there is considerable uncertainty regarding the exact development in future life expectancy. This uncertainty is called systematic longevity risk.[1] Systematic longevity risk can be modeled by allowing future survival probabilities to be stochastic. In studies investigating optimal annuity decisions systematic longevity risk is typically ignored. However, its presence affects the optimal life-cycle decisions in a number of ways. First, systematic longevity risk is a non-diversifiable risk and therefore it will have a nonzero price of risk, complicating the pricing of annuities. Second, stochastic future survival probabilities imply stochastic future annuity prices, further complicating the optimal life-cycle decisions of the individual. We allow for a nonzero price of risk in the annuity prices using risk-neutral survival probabilities, following Cairns, Blake, and Dowd (2006a). Third, we allow for stochastic future survival probabilities and optimal decisions which depend on

[1] Naturally, we also allow for idiosyncratic longevity risk, which is due to a random individual remaining lifetime, conditional on given survival probabilities. This is also referred to as non-systematic longevity risk.

the evolution of these survival probabilities. To solve the optimization problem with stochastic future survival probabilities we follow Brandt, Goyal, Santa-Clara, and Stroud (2005) and Carroll (2005), and using extensions proposed by Koijen, Nijman, and Werker (2009). In addition to allowing for systematic longevity risk,[2] we extend the current literature on optimal annuity decisions by not only investigating the possibility of investing in an immediate annuity but also in a deferred annuity.

The existing literature on optimal annuitization in the context of immediate annuities but without systematic longevity risk is extensive. The literature was initiated by the seminal paper by Yaari (1965). Yaari (1965) and others (see, for example, Merton, 1983; and Davidoff, Brown, and Diamond, 2005) show that an individual's optimal investment choice is to invest all his wealth in annuities. This is shown in a standard Modigliani life-cycle model of savings and consumption without a bequest motive, with as only investment opportunity a risk-free asset and actuarially fair annuities. The rationale behind this result is that the returns from annuities dominate the risk-free return, since the capital invested in annuities is allocated only to the survivors. Although these results suggest that retirees will voluntarily purchase annuities, in most countries very few actually do so (see, among others, Friedman and Warshawsky, 1990; Poterba and Wise, 1998; Moore and Mitchell, 2000; Bütler and Teppa, 2007; and Dushi and Webb, 2004b). This "annuity puzzle" has generated a lot of literature aimed at solving this puzzle.[3] We show that systematic longevity risk reduces the attractiveness of an immediate annuity, thereby reducing the optimal level of annuitized wealth when purchasing an annuity at retirement date. Thus, systematic longevity risk seems to be an important ingredient in understanding the annuity puzzle.

This chapter also contributes to the literature on the optimal timing of the

[2]Cocco and Gomes (2009) also allow for systematic longevity risk in a life-cycle model, but in their paper the individual maximizes the expected lifetime utility in a setting with only a risk-free asset and a longevity bond, without annuities or equities as investment opportunities.

[3]For example, the evidence for the size of an individual's bequest motive and the corresponding effect in a life-cycle model is mixed (see, for example, Yaari, 1965; Friedman and Warshawsky, 1990; Bernheim, 1991; Brown and Poterba, 2000; Hurd and Smith, 2001; and Vidal-Meliá and Lejárraga-García, 2006). Other authors have examined the strategic bequest motive (see, for example, Bernheim, Shleifer, and Summers, 1985). Another possible explanation for the annuity puzzle is the default risk of the annuity issuer (see Babbel and Merrill, 2007) or the illiquidity or irrevocability of annuities (see Sinclair and Smetters, 2004; and Peijnenburg, Nijman, and Werker, 2009). In addition, behavioural effects (see, for example, Hu and Scott, 2007; Brown, 2007; Brown, Kling, Mullainathan, and Wrobel, 2008; and Gazzale and Walker, 2009) may influence the decision to forgo voluntary annuitization.

purchase of annuities. Much research has been devoted to finding the optimal fraction of wealth invested in annuities and the best timing for purchasing annuities. Due to actuarial unfairness of annuities postponing the purchase of an annuity purchase may be rational, because the mortality credit is too low just after retirement age.[4] Milevsky (1998) proposed postponing the annuity purchase until the mortality credit is larger than or equal to the equity risk premium. However, this annuitization strategy would only be optimal for risk-neutral individuals. Blake, Cairns, and Dowd (2003) found an optimal annuitization age in the range of 65 to 80, depending on individual characteristics such as risk aversion and bequest motive. Milevsky and Young (2002) estimated that the real option to annuitize remains valuable until the age range 75-85, also depending on individual characteristics. Different assumptions of the utility function have been made to find the optimal time to purchase annuities. These include the HARA utility (see Kingston and Thorpe, 2005) and the power utility (see Stabile, 2006). Others have investigated the optimal gradual annuity purchase pattern during retirement. For example, Kapur and Orszag (1999) and Horneff, Maurer, and Stamos (2008) found that gradual annuitization is optimal until the mortality credit is larger than the equity return. In our setting we find that for a 65-year-old individual postponing the annuity purchase is utility-decreasing due to systematic longevity risk. This difference with the existing literature illustrates the importance of systematic longevity risk in the life-cycle optimization problem.

This chapter also contributes to the literature on the attractiveness of deferred annuities. The literature on the optimal deferral period of a deferred annuity which is purchased at retirement date is not very extensive. Milevsky (2005) provides a description of (inflation-linked) deferred annuities, referred to as Advanced-Life Deferred Annuities (ALDAs). This chapter states that deferred annuities, purchased at the retirement date, starting to pay after a deferral period of around 15 to 25 years, are optimal. A deferred annuities is optimal because such an annuity provides longevity insurance at a low price. Hu and Scott (2007) mention that deferred annuities may be more desirable for individuals than immediate annuities, because the former overweight small probabilities. Dus, Maurer, and Mitchell (2005) show that deferred annuities can enhance the expected payout and cut the expected shortfall risk. In a setting with only a risk-free asset and given rules of thumb for the consumption level, Gong and Webb (2009) show that deferred annuities provide

[4] The mortality credit is defined as the (yearly) excess return of an annuity relative to the return on the risk-free investment. The mortality credit is formally defined in Section 6.4.

longevity insurance at a low cost. Horneff and Maurer (2008) find that a deferred annuity which starts income payments at the age of 65 might become more appealing than an immediate one when the loading factor is high enough. Bayraktar and Young (2009) find that it is always optimal to purchase an immediate annuity instead of a deferred one, when an individual's objective is to minimize the probability of financial ruin. We extend this literature in two ways. First, we compare annuities with different deferral periods. Second, we take into account that the actuarial unfairness in annuity prices may be due to a risk premium for systematic longevity risk. We use a risk-neutral pricing approach to model the risk premium in annuities due to systematic longevity risk. This results in annuities with a risk premium for systematic longevity risk that is dependent on the deferral period. We find that the optimal deferral period is short and that the utility gain from purchasing an annuity with the optimal deferral period instead of an immediate annuity is very small. Moreover, we find that when an individual purchases an annuity with a moderate deferral period (around 10 years) he can hold a substantial amount of liquid wealth with a low reduction in the expected lifetime utility.

This chapter is organized as follows. In Section 6.2 we present the preferences of the representative individual and describe the stochastic forecast models we use to forecast the probability distribution of future survival probabilities. Section 6.3 presents the parameter calibration of the distributions of the equity returns and the distribution of the future survival probabilities. The attractiveness of an annuity is affected by the price and the payment stream. Therefore, in Section 6.4 we first illustrate the effect of systematic longevity risk on both the price of a deferred annuity and an immediate annuity. In addition, we illustrate the effect of systematic longevity risk on the mortality credit. In Section 6.5 we determine the optimal choices of an individual in the expected lifetime utility model. We show the effect of different annuity choices and the effect of systematic longevity risk on the optimal decisions of the individual. Robustness checks are subsequently performed in Section 6.6. Section 6.7 presents the conclusions.

6.2 Preferences, survival probabilities, and annuities

This chapter investigates an individual's optimal fraction of wealth invested in either a deferred annuity or an immediate annuity, and the effect of systematic longevity risk on this decision. The optimal annuity decision is determined in a setting with three sources of risk:

i) *investment risk*, caused by a random return in the equity market;

ii) *idiosyncratic longevity risk*, due to a random individual remaining lifetime (conditional on given survival probabilities);

iii) *systematic longevity risk*, due to random future survival probabilities.

Section 6.2.1 defines an individual's expected lifetime utility function and describes the constraints the individual faces. The optimal choices of an individual depend on the probability distribution of future survival probabilities which is described in Section 6.2.2, and on the pricing of annuities, which is described in Section 6.2.3.

6.2.1 The individual's optimization problem

In this section we describe the optimization problem including the constraints faced by an individual who maximizes an expected utility of lifetime consumption. The investment choice consists of the fractions of wealth invested in a risky asset, a risk-free asset, and in an annuity. We consider two types of annuities:

i) An *immediate annuity* which yields a nominal yearly payment of 1, with a final payment in the year the insured dies;

ii) a *deferred annuity* which yields a nominal yearly payment of 1, after a deferral period of d years when the insured is still alive, with a final payment in the year the insured dies.

Let $A^{(d)}_{x+t,t}$ denote an annuity with a deferral period of d years bought by an individual at time t with time-0 age x . Note that an immediate annuity represents a special type of a deferred annuity, namely one with a deferral period equal to one (i.e., $d = 1$).[5] We determine an individual's lifetime expected utility and optimal choices

[5]An annuity can either be an *ordinary annuity* or an *annuity-due*. The difference between the two types of annuities is that an annuity income payments can either be at the beginning of a specified period (i.e., an annuity-due) or at the end of the specified period (i.e., an ordinary annuity). An annuity with $d = 1$ is an ordinary immediate annuity and an annuity with $d = 0$ is an immediate annuity-due. Note that using only a deferred annuity with a deferral period of zero years, one can obtain the same payment stream as with a deferred annuity with a deferral

for two cases, namely currently purchasing a deferred annuity with a fixed deferral period d and postponing the purchase of an immediate annuity until a fixed time s.[6]

We assume that the individual has an intertemporally separable, expected lifetime constant relative risk aversion (CRRA) utility function, without a bequest motive. To avoid overloaded notation, the time at which we calculate the expected lifetime utility, i.e., the base year, is set equal to zero, unless otherwise mentioned. We assume that the individual invests in an annuity only once, at a fixed time $s \geq 0$, and invests in only one type of annuity, i.e., an annuity with deferral period d (with $d = 1$ for an immediate annuity). To investigate the effect of the different types of annuities and the effect of postponing the annuity purchase we calculate the optimal consumption, investment, and annuity choices conditional on the annuity type (i.e., conditional on d), and the time when an annuity is purchased (i.e., conditional on s). We obtain the optimal annuity choice by comparing the corresponding expected lifetime utilities.

We consider an individual aged x at date $t = 0$, with remaining lifetime $T_{x,0}$, and with corresponding *realized* survival probabilities $p_{x+s,s}$ as defined in (2.4) for $t = 0$. Note that at any date $t \geqslant 0$, it holds true that

$$\begin{aligned} {}_{\tau}p_{x+t,t} &= P\left(T_{x,0} \geqslant t+\tau | T_{x,0} \geqslant t, \mathcal{F}_{\infty}\right) \\ &= \frac{P\left(T_{x,0} \geqslant t+\tau | \mathcal{F}_{\infty}\right)}{P\left(T_{x,0} \geqslant t | \mathcal{F}_{\infty}\right)} \\ &= \prod_{s=t}^{t+\tau-1} p_{x+s,s}. \end{aligned} \tag{6.1}$$

Let γ denote the coefficient of relative risk aversion; let β be the time preference parameter (also referred to as the subjective discount factor); let W_t be the liquid wealth level in period t; let C_t be the consumption level in period t; and let A_t be the annuity income in year t. An individual is characterized by his time-0 age x, wealth level before annuity income and consumption, W_t, and the time-t state variables corresponding to the annuity income, A_t, and B_t.

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period of one year. This occurs because the difference between the two annuities is only a certain immediate payment. Under arbitrage-free pricing the price of an annuity with an initial payment in the following year and an annuity with an immediate initial payment equals the level of the current payment. In this chapter, when we refer to immediate annuities, we mean an ordinary annuity with an initial payment in the following year, i.e., $d = 1$.

[6] Note that one can also investigate the effect of postponing the purchase of deferred annuities. However, as we will argue in Section 6.7, this will probably not be optimal due to the systematic longevity risk.

Now we consider a given time s at which annuities with a deferral period of d years are bought, and determine the optimal investment and consumption choices. At time t the endogenous state variables are W_t, A_t, and B_t and the exogenous state variables at time t are denoted by the vector X_t.[7] Let (C_t, w_t) be the set of control variables in year t, i.e., the time-t level of consumption and the fraction of wealth after annuity income and consumption invested in equity, respectively, and let $a_s(d)$ be an additional control variable at time s, i.e., the fraction of after-consumption wealth which in year s is invested in an annuity with a deferral period of d years. The time-t expected lifetime utility J_t of an individual is defined by:

$$\begin{aligned} J_t\left(W_t, A_t, B_t, X_t\right) &= \max_{a_s(d), \{w_\tau, C_\tau\}_{\tau \geq t}} \left\{ \mathbb{E}_t \left[\sum_{\tau \geq 0} \prod_{s=t}^{t+\tau-1} p_{x+s,s} \beta^\tau \frac{(C_{t+\tau})^{1-\gamma}}{1-\gamma} \right] \right\}, \quad \text{if } t \leq s \\ &= \max_{\{w_\tau, C_\tau\}_{\tau \geq t}} \left\{ \mathbb{E}_t \left[\sum_{\tau \geq 0} \prod_{s=t}^{t+\tau-1} p_{x+s,s} \beta^\tau \frac{(C_{t+\tau})^{1-\gamma}}{1-\gamma} \right] \right\}, \quad \text{if } t > s. \end{aligned} \tag{6.2}$$

At or before time s, the individual maximizes his expected lifetime utility with the fraction of liquid wealth invested in equity, the consumption, and the fraction of wealth invested in annuities at time s as control variables. After time s the individual does not purchase new annuities, and hence the control variables are only the sequence of current and future fractions of liquid wealth invested in equities, and the yearly consumption levels.

The wealth dynamics of the individual, for all $\tau \geq 0$, are given by:

$$W_{\tau+1} = \begin{cases} (W_\tau - C_\tau) \cdot (1 - a_s(d)) \cdot \left(1 + r^{rf} + w_\tau \cdot \left(r_\tau - r^{rf}\right)\right), & \text{if } \tau = s, \\ (W_\tau + A_\tau - C_\tau) \cdot \left(1 + r^{rf} + w_\tau \cdot \left(r_\tau - r^{rf}\right)\right), & \text{if } \tau \neq s, \end{cases} \tag{6.3}$$

where r^{rf} is the time-independent risk-free return, and r_τ is the (risky) return on equity between year τ and $\tau + 1$. The first equation corresponds to the wealth dynamics in the year in which the individual purchases an annuity and the second corresponds to the wealth dynamics in the years in which the individual does not purchases an annuity. The individual faces a sequence of short-selling constraints and liquidity constraints. These constraints imply that an individual cannot borrow against future income. Hence, the objective function for the individual, as

[7] The exogenous state variables depend on the evolution of the future survival probabilities up to time t. The evolution of the survival probabilities is described in Section 6.2.2. The evolution of an individual's information about the distribution of the future survival probabilities is described in Appendix 6.B.

represented in (6.2), is maximized subject to the wealth dynamics in (6.3) and the following constraints:

$$0 \leq a_s(d) \leq 1, \tag{6.4}$$

$$0 \leq w_\tau \leq 1, \qquad \text{for } \tau \geq 0, \tag{6.5}$$

$$C_\tau \leq W_\tau + A_\tau, \qquad \text{for } \tau \geq 0. \tag{6.6}$$

Equations (6.4)–(6.5) correspond to the no short-selling constraints, and equation (6.6) implies that the individual cannot borrow against future income. Given that the individual purchases an annuity at time s with a deferral period of d years, the annuity income level is, by definition, given by:

$$A_{t+1} = \begin{cases} A_t, & \text{if } t \neq s+d-1, \\ B_t, & \text{if } t = s+d-1, d > 1, \\ \frac{a_s(1)\cdot(W_s - C_s)}{V_s\left(A^{(1)}_{x+s,s}\right)}, & \text{if } t = s, d = 1, \end{cases} \tag{6.7}$$

with $A_0 = 0$, $V_s\left(A^{(1)}_{x+s,s}\right)$ the time-s price of an immediate annuity, and

$$B_{t+1} = \begin{cases} B_t, & \text{if } t \neq s, \\ \frac{a_s(d)\cdot(W_s - C_s)}{V_s\left(A^{(d)}_{x+s,s}\right)}, & \text{if } t = s, \end{cases}$$

with $B_0 = 0$, and $V_s(A^{(d)}_{x+s,s})$ the time-s price of an annuity with a deferral period of d years. The state variable B_t does not play a role when $d = 1$.

To obtain the optimal consumption and investment choices we use a simulation-based method which can deal with many exogenous state variables, proposed by Brandt, Goyal, Santa-Clara, and Stroud (2005) and by Carroll (2006). In addition, we include several extensions which were proposed by Koijen, Nijman, and Werker (2009). In Appendix 6.B the method used to obtain the optimal consumption and investment choices is described.

6.2.2 Survival probabilities

As can be observed from equation (6.2) the optimal life-cycle choices of an individual depend on future survival probabilities of the individual. In this chapter future survival probabilities are stochastic. In this section we describe the modeling of the probability distribution of future survival probabilities. We use the model proposed in Cairns, Blake, and Dowd (2006a). This CBD model is attractive because it uses only a few parameters to obtain a good fit of the mortality probabilities and it has already been extended to include a market price of systematic longevity risk in an empirically justified method. In the CBD model the mortality curve is a special case of the Perks models (see, for example, Perks, 1932, and Benjamin and Pollard, 1993). Let $q_{x+t,t} = 1 - p_{x+t,t}$ be the *realized* one-year death probability for the cohort aged x at time 0. In the CBD model the logit of the one-year mortality probability is modeled as:

$$\log\left(\frac{q_{x+t,t}}{1-q_{x+t,t}}\right) = k_t^{(1)} + (x+t)\cdot k_t^{(2)} + \epsilon_{x+t,t}, \tag{6.8}$$

where $k^{(i)} = \left[k_{\underline{t}}^{(i)}, k_{\underline{t}+1}^{(i)}, \ldots\right]'$ for $i \in \{1,2\}$ are stochastic processes with $\underline{t}$ being the first year of mortality data, and $\epsilon_{x+t,t}$ an the age- and time-specific idiosyncratic residual assumed to be independent and identically distributed (i.i.d.) normally distributed with zero mean and age-specific variance.

We estimate the process for mortality probabilities using mortality data. Let $\bar{t}$ be the last year of the mortality data. Then, in order to project the future mortality probabilities the individual needs the future values of the stochastic processes $k_t = \left[k_t^{(1)}\ k_t^{(2)}\right]'$, for $t > \bar{t}$. Following Cairns, Blake, and Dowd (2006a) we assume that the individual forecasts these stochastic processes by a two-dimensional random walk with drift:

$$k_{t+1} = k_t + \mu + C\cdot N_t, \tag{6.9}$$

where μ is a constant 2×1 vector, C is a constant 2×2 upper triangular matrix, and N_t is a two-dimensional standard Gaussian process. For longevity risk it is common to include not only process risk, i.e., the risk arising from the random process N_t, but also parameter risk (see, for example, Cairns, Blake, and Dowd, 2006a). There is a consensus in the literature that the exclusion of parameter uncertainty, given a specification like (6.8)–(6.9), would lead to a significant underestimation of longevity uncertainty.[8] Let $D_t = k_t - k_{t-1}$, and $D = [D_{\underline{t}+1}, \ldots, D_{\underline{t}+n}]'$ with $n = \bar{t} - \underline{t}$. To incorporate parameter risk in the parameters μ and $V = C\cdot C'$ we use the Jeffrey's prior as in the CBD model, a non-informative prior distribution, which is a common prior for the multivariate Gaussian distribution in which both μ and V are unknown:

$$p(\mu, V) \propto |V|^{-3/2},$$

[8] Likewise, the investor might allow for parameter uncertainty in the equity process. Since our focus in this chapter is on the effect of systematic longevity risk, we assume that the investor only accounts for process risk in the equity process.

where $|V|$ is the determinant of the covariance matrix V. The posterior distribution for $(\mu, V|D)$ at time τ satisfies:

$$V^{-1}|D \sim \text{Wishart}\left(\tau - 1, \tau^{-1}\widehat{V}_{\tau}^{-1}\right), \tag{6.10}$$

$$\mu|V, D \sim MVN\left(\widehat{\mu}_{\tau}, \tau^{-1}V\right), \tag{6.11}$$

where $\widehat{\mu}_{\tau}$ and $\widehat{V}_{\tau}$ are the maximum likelihood estimates of the parameters of the stochastic process based on the revealed information up to time τ. At any time $\tau > \underline{t}$ these parameters are estimated by:

$$\widehat{\mu}_{\tau} = \frac{1}{\tau - \underline{t}} \cdot \sum_{t=\underline{t}+1}^{\tau} D_t, \tag{6.12}$$

$$\widehat{V}_{\tau} = \frac{1}{\tau - \underline{t}} \cdot \sum_{t=\underline{t}+1}^{\tau} \left(D_t - \widehat{\mu}_{\tau}\right) \cdot \left(D_t - \widehat{\mu}_{\tau}\right)'. \tag{6.13}$$

6.2.3 Annuity prices

The price of an annuity depends on the probability distribution of an individual's remaining lifetime. The actuarially fair price (i.e., the expected discounted cash flows) of an annuity can be determined using the probability distribution of the stochastic future survival probabilities which is described in Section 6.2.2. The actuarially fair value of an annuity does not incorporate the effect of systematic longevity risk on the price the annuity. Idiosyncratic longevity risk is diversifiable (i.e., the risk becomes negligible when the the portfolio size is sufficiently large) and thus will not be priced in an efficient market without arbitrage opportunities. In contrast, systematic longevity risk does not decrease with the portfolio size, and may thus lead to a risk premium.

There is strong empirical evidence that the market price of annuities exceeds the actuarially fair one (see Mitchell, Poterba, Warshawsky, and Brown (1999) for the US market and Frinkelstein and Poterba (2002) for the UK market). To account for this, the existing life-cycle model literature commonly uses a loading factor. Commonly, the loading factor is seen as a transaction cost, (see, among others, Mitchell et al., 1999). However, the actuarial unfairness might be due to the price of systematic longevity risk rather than transaction costs, which is also mentioned in Milevsky and Young (2007). The premium for systematic longevity risk in an annuity might be significant which implies that the actuarial fair value might be a significant underestimation of the market price.[9] More importantly, a loading factor independent of the deferral period of an annuity does not necessarily properly reflect the risk premium for systematic longevity risk.

No liquid market exists yet for systematic longevity risk (see Blake, Cairns, and Dowd, 2008). Therefore, it is difficult to calibrate the market price of systematic longevity risk. The existing literature proposes different approaches to obtain a fair value for annuities when systematic longevity risk exists. These approaches include the utility maximization principle (see Malamund, Trubowitz, and Wüthrich, 2008); the Sharpe ratio approach (see, for example, Milevsky, Promislow, and Young, 2006, 2008; Bayraktar, Milevsky, Promislow, and Young, 2009; and Bauer, Börger, and Ruß, 2009); the Wang transform (see Lin and Cox, 2005; Cox and Lin, 2007; Denuit, Devolder, and Goderniaux, 2007; and Lin and Cox, 2008), and risk-neutral pricing (see Cairns, Blake, and Dowd, 2006a). An excellent overview of different pricing methods is given in Bauer, Börger, and Ruß (2009).

In this chapter we use the risk-neutral approach to calculate the risk premium for systematic longevity risk. The risk-neutral approach is based on long-established financial economic theory and states that, if the overall market is arbitrage-free, there exists a risk-neutral measure such that the price of an annuity equals the expected discounted payments under the risk-neutral measure. Due to market incompleteness many risk-neutral risk measures might exist. Therefore, we shall assume that an individual is acting in an equilibrium setting, and that this equilibrium selects a market consistent (unique) risk neutral measure. Following the existing literature on life-cycle models we also use, as alternative to the risk-neutral approach, a loading factor for pricing annuities. As loading factor we take 7.3%, which is in line with Mitchell et al. (1999), and commonly used in the life-cycle literature (see, for example, Horneff, Maurer, and Stamos, 2008). Hence, this chapter considers annuity market prices that exceed the actuarially fair ones which are modeled using:

i) *risk-neutral survival probabilities*;

ii) a *loading factor*, which is independent of the deferral period of an annuity.

[9] For example, the Solvency II project (Group Consultatif Actuariel Europeen, 2008) requires the valuation of annuities using a market-to-model approach. The approach proposed in the Solvency II guidelines leads to a valuation of an annuity which is approximately 6% to 9% higher than the real-world expected discounted cash flows of an annuity for an annuitant aged 65, due to systematic longevity risk (see Olivieri and Pitacco, 2008b).

Especially for deferred annuities the method used to price annuities is important. Using our risk-neutral survival probabilities the actuarial unfairness (measured by the fraction of the price of an annuity due to the risk margin for systematic longevity risk) in deferred annuities is an increasing function of the deferral period. In contrast, using a loading factor the actuarial unfairness is independent of the deferral period. To illustrate that the results are robust for these different types of pricing annuities, we will use both methods (i.e., the risk-neutral pricing method and the loading factor) to price annuities.

Let us now describe the risk-neutral method to obtain the price of an annuity. Recall that ${}_{\tau}p_{x+t,t}$ is the probability that an $x+t$-year old at time t will survive at least τ years, and r^{rf} is the risk-free return. Let $\mathbb{E}_t^{\mathcal{Q}}[\cdot]$ denote the time-t expectation under the risk-neutral measure. Then the time-t market price of an annuity $V_t(A_{x+t,t}^{(d)})$ is given by:[10]

$$V_t(A_{x+t,t}^{(d)}) = \mathbb{E}_t^{\mathcal{Q}}\left[\sum_{\tau\geq d} 1_{\tau}^{(x+t,t)} \cdot \left(\frac{1}{1+r^{rf}}\right)^{\tau}\right] = \sum_{\tau\geq d} \mathbb{E}_t^{\mathcal{Q}}\left[{}_{\tau}p_{x+t,t}\right] \cdot \left(\frac{1}{1+r^{rf}}\right)^{\tau}, \tag{6.14}$$

where $1_{\tau}^{(x+t,t)}$ is an indicator which equals one if the individual with age $x+t$ at time t is alive at time $t+\tau$, and zero otherwise.

As can be observed from equation (6.14), to calculate the price of an annuity using the risk-neutral approach we need the risk-adjusted expectation of future survival probabilities. The probability distribution of future survival probabilities is described in Section 6.2.2. To include the market price of systematic longevity risk we follow the method proposed in Cairns, Blake, and Dowd (2006a). In this method the risk-adjusted pricing measure $\mathcal{Q}(\lambda)$ is modeled using an adjustment in the dynamics of the stochastic process k_t. Let $\lambda = [\lambda_1\ \lambda_2]'$ be the vector representing the market price of systematic longevity risk, which is assumed to be time-independent. The dynamics of the process k_t under the real-world measure is described in equation

[10] In case of uncertainty in the instantaneous forward interest rate curve it has been shown that, under the assumption of independence of the instantaneous forward interest rate and survival probabilities, equation (6.14) still holds, replacing $\left(\frac{1}{1+r^{rf}}\right)^{\tau}$ on the right hand by the time-t price of a zero-coupon bond with face value 1 maturing in year $t+\tau$ (see, for example, Cairns, Blake, and Dowd, 2006a).

(6.9). The process $\widetilde{k}_t$ under the risk-adjusted measure $Q(\lambda)$ is given by:

$$\begin{aligned}\widetilde{k}_{t+1} =& \widetilde{k}_t + \mu + C \cdot (N_t - \lambda), \\ =& \widetilde{k}_t + \widetilde{\mu} + C \cdot N_t,\end{aligned} \tag{6.15}$$

where $\widetilde{\mu} = \mu - C \cdot \lambda$. Whereas the individual updates the parameters μ and V continuously, we assume that the parameter λ is not updated over time.

6.3 Parameter calibration

6.3.1 Financial market

In this section we describe the financial market return processes. We assume that, besides different types of annuities, the financial market consists of a risk-free asset, and a risky asset. The yearly return on the risk-free asset is set at 4% and assumed to be time-independent. Hence, we have that $r^{rf} = 0.04$. Define S_t as the time-t stock price, assuming that there are no dividends.[11] Then $r_t \equiv \frac{S_{t+1}}{S_t} - 1$ is the yearly equity return between year t and $t+1$. The stock price is modeled as a Brownian motion with drift:

$$dS_t = \mu^S \cdot S_t \cdot dt + \sigma^S \cdot S_t \cdot dZ_t,$$

where $\mu^S \equiv r^{rf} + \lambda^S \cdot \sigma^S$ and σ^S are model parameters, with λ^S the parameter for the market price of equity risk, and Z_t is a standard Brownian motion. Following the life-cycle literature we set λ^S equal to 0.155 and σ^S equal to 0.158, resulting in an expected yearly excess equity return equal to 4% and a standard deviation of the yearly equity return equal to 17% (see, for example, Gomes and Michaelides, 2005; Yao and Zang, 2005; and Cocco, Gomes, and Meanhout, 2005). The equity risk premium of 4% is lower than the historical one, which is very common in this literature. The lower-than-historical return is an adjustment in order to take transaction costs into account, most of which are in the form of mutual fund fees. Due to the high dimensionality of modeling the transaction costs explicitly (as is, for example, done in Heaton and Lucas, 1996) it is common to use this shortcut representation of a lower expected return to take into account transaction costs. In Section 6.6 as a robustness check we set λ^S equal to 0.343 resulting in an expected

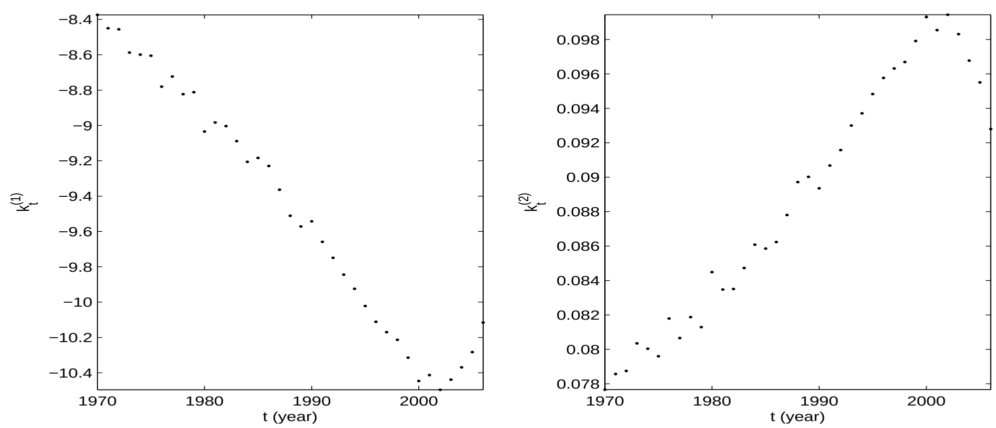
[11]Note that when there are dividends, the return dynamics are not affected when the dividends are reinvested in equities.

yearly equity excess return equal to 7%. This leads to an equity risk premium which is in line with historical results (see, for example, Brennan and Xia, 2002).

6.3.2 Systematic longevity risk

To estimate the parameter values of the stochastic processes $k^{(1)}$ and $k^{(2)}$ we use age-, gender-, and time-specific mortality probabilities for the United States, obtained from the Human Mortality Database.[12] Figure 6.3.1 displays the parameter values of the stochastic process for males in the US from $\underline{t} = 1970$ to $\overline{t} = 2006$, by fitting (6.8). The value of $k_t^{(1)}$ displays the time-t general level of mortality. The general decrease over time in the level of the stochastic process $k_t^{(1)}$ implies that there is generally a decrease in the level of the mortality probabilities over time. The general increase over time in the value of $k_t^{(2)}$ implies that the mortality reduction is generally lower at higher ages.

Figure 6.3.1: **Estimated parameter values of the stochastic processes k**



This figure displays the estimated parameter values of the stochastic processes $k^{(1)}$ and $k^{(2)}$. The left panel displays the estimated parameter values of the stochastic processes $k^{(1)}$, the right panel displays the estimated parameter values of the stochastic processes $k^{(2)}$. The stochastic processes are estimated using US male mortality data from 1970 to 2006.

Using US male mortality probabilities from 1970 to 2006 we obtain the esti-

[12]Available from the Human Mortality Database: www.mortality.org.

mates of the parameters in equations (6.12) and (6.13):

$$\widehat{\mu}_{\overline{t}} = \begin{bmatrix} -0.048383 \\ 0.00042065 \end{bmatrix}$$
$$\widehat{V}_{\overline{t}} = \widehat{C}_{\overline{t}} \cdot \widehat{C}'_{\overline{t}} = \begin{bmatrix} 0.0069237 & -0.00010012 \\ -0.00010012 & 1.4765 \cdot 10^{-6} \end{bmatrix},$$

where $\widehat{C}_{\overline{t}}$ can be recovered from a Choleski decomposition of $\widehat{V}_{\overline{t}}$.

Recall that the base year is set equal to 0. To forecast the distribution of the future mortality probabilities we take as the starting value of k_0 the estimate $\widehat{k}_{\overline{t}}$ corresponding to $\overline{t} = 2006$, i.e., $k_0 = \widehat{k}_{\overline{t}} = [-10.1157\ 0.092799]'$. The mortality probabilities are forecasted by simulating the parameters μ and V for each path. Let MA be the maximum attainable age, which is set at 110 years. Then, given the simulated parameters μ and V, we simulate the paths of k_t for each future time period, i.e., for $t = 1, \ldots, MA - x$, given the assumed values of k_0. The distribution of future mortality probabilities is obtained by the simulated values of the stochastic processes k_t and the simulated residuals $\{\epsilon_{x+t,t} | x + t \in \{x, \ldots, MA\}, t \in \{1, \ldots, MA - x\}\}$. Each such path gives us a path of the future mortality probabilities.

6.3.3 Pricing systematic longevity risk

In Section 6.2.3 we mentioned that the market price of an annuity exceeds the actuarially fair one, which might be due to systematic longevity risk. Since there is no liquid market for systematic longevity risk it is difficult to calibrate the risk-neutral survival probabilities using empirical data. Although little information is available, it is reasonable to assume that life insurers requires a positive risk premium for systematic longevity risk.[13] This implies that the physical expectation of the discounted cash flows is lower than the risk-adjusted expectation.

The risk adjusted process for survival probabilities, as defined in equation (6.15), depends on the parameter λ. Cairns, Blake, and Dowd (2006a) calibrated as parameter value $\lambda = [0.175\ 0.175]'$, using the EIB/BNP longevity bond, which was

[13]Since there is no market for systematic longevity life insurers, there is no equilibrium price for it, and insurers would have to hold a reserve for systematic longevity risk. There exists also natural counterparties of systematic longevity risk in annuities, for example death benefit providers. However, it is still likely that there is a positive risk premium, because short (for example annuities) exposure is 30-40 larger than long (for example death benefits) exposure (source: American Council of Life Insurers, "U.S. Life Insurance Moody's Statistical Handbook", August 2006; "Pension Markets in Focus", OECD, October 2006; and Moody's U.K. Life Insurance Industry Outlook, January 2007).

announced in November 2004.[14] We will use this calibrated value of λ for pricing annuities.[15] Using the calibrated parameter $\lambda = [0.175\, 0.175]'$ in the risk-neutral pricing we obtain a market price for an immediate annuity which is 7.35% higher than the actuarially fair one. Using the empirical prices of immediate annuities Mitchell et al. (1999) found a loading factor of 7.3%.[16] This might imply that using only the risk-adjusted process $Q(\lambda)$ with $\lambda = [0.175\, 0.175]'$, one could explain the price observed in the real-world by only using the risk-neutral pricing method for systematic longevity risk.

One might argue that the price of systematic longevity risk using the calibrated parameter from Cairns, Blake, and Dowd (2006a) (i.e., $\lambda = [0.175\, 0.175]'$) might be an overestimation of the real one, since the longevity bond was withdrawn prior to issue. As an alternative, we assume that the individual postulates a uniform distribution on $\lambda = [\lambda_1\, \lambda_2]'$, i.e., $\lambda_1 = \lambda_2 \sim U\,(0, 0.175)$. Note that this stochastic λ leads to a lower risk premium for systematic longevity risk than the calibrated $\lambda = [0.175\, 0.175]'$.

In summary, for the pricing of annuities we distinguish three cases:

i) the calibrated parameter on the EIB/BNP longevity bond, i.e., $\lambda = [0.175\, 0.175]'$;

ii) a stochastic λ, with $\lambda_1 = \lambda_2 \sim U\,(0, 0.175)$;

iii) a loading factor, which is set equal to 7.3%, irrespective of the deferral period.

The last case is included in order to compare our results with the existing literature on life-cycle models. As we will show in the following section, for deferred annuities the risk-neutral pricing method implies that the risk premium as fraction of he

[14]The EIB/BNP longevity bond was withdrawn prior to issue. One of the reasons why this issue was unsuccessful might be the price of the longevity bond indicating that this calibrated λ overestimates the price of systematic longevity risk. However, there are several design issues (see Blake, Cairns, and Dowd (2008) for an extensive investigation of the failure of this longevity bond) which might explain why the bond was withdrawn prior issue.

[15]The longevity bond was based on publicly available Office for National Statistics (ONS) data on English and Welsh mortality for a cohort of males aged 65 in 2003. Tuljapurkar, Nan, and Boe (2000), among others, have shown that the mortality development in western countries has similar patterns. This indicates that that the driving forces for the decline in mortality may be the same in western countries, which implies that the price of longevity risk would be similar for western countries. Hence, it might indicate that the market price of risk (λ) is approximately the same in western countries.

[16]This holds for immediate annuities. Since we do not have information on the loading factor for deferred annuities we do not know whether this also holds for deferred annuities.

actuarially fair price of an annuity premium is increasing in the deferral period. By determining the optimal choices in the life-cycle model in a setting where annuities are priced using a constant loading factor we show that our results also hold when the actuarial unfairness in annuities is independent of the deferral period of a deferred annuity.

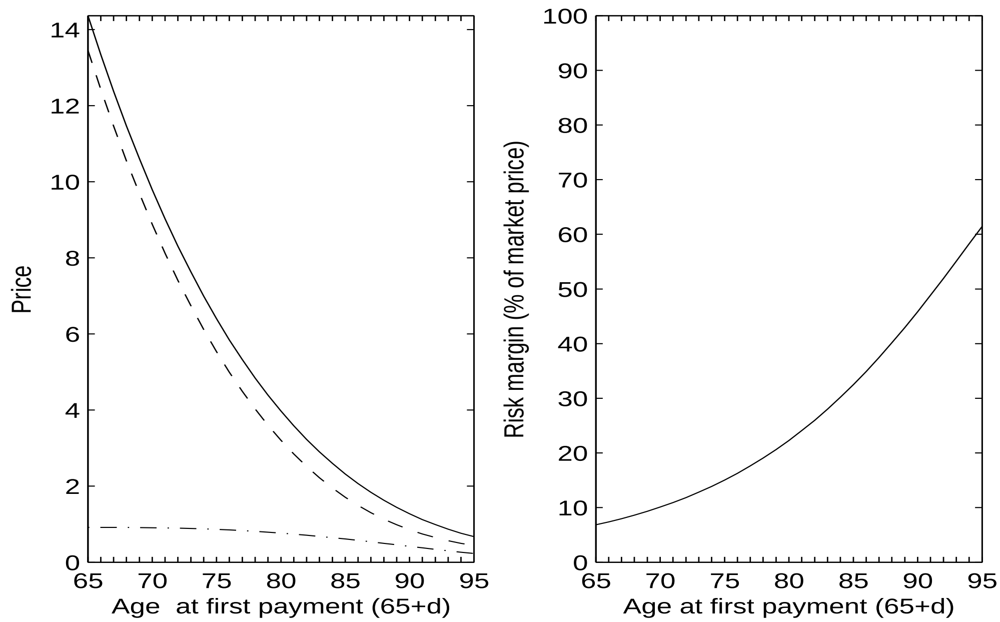
6.4 The effect of longevity risk on annuity prices

This chapter extends the existing literature on life-cycle models in two ways, namely by including systematic longevity risk and by allowing an individual to purchase a deferred annuity. Clearly, the optimal annuity decision also depends on the current and future prices of the different annuities. Therefore, before investigating the optimal annuity decision in a utility framework setting in Section 6.5, we investigate in this section the effect of systematic longevity risk on the price of immediate and deferred annuities.

Systematic longevity risk affects the price of annuities in two ways. First, systematic longevity risk leads to a risk premium in the price of annuities. Second, it leads to uncertain future survival probabilities and thus to stochastic future annuity prices. We determine the current price of deferred annuities and the probability distribution of the price of immediate annuities purchased at time $s > 0$, using the model to forecast the distribution of future survival probabilities. For illustrative purposes we consider a 65-year-old male at time $t = 0$ and set $\lambda = [0.175\ 0.175]'$ (see Section 6.3.3).

Figure 6.4.1 displays the effect of systematic longevity risk on the current price of deferred annuities, as a function of the deferral period. The left panel displays the price as a function of the deferral period (solid curve, the expectation under the $\mathcal{Q}$-measure), the actuarially fair price (dashed curve, the expectation under the $\mathcal{P}$-measure), and the risk premium for the systematic longevity risk of the annuity (dashed-dotted curve, the difference in the expectation under the $\mathcal{Q}$- and the $\mathcal{P}$-measure). The right panel displays the fraction of the price of the annuity which is due to the risk premium, i.e., the risk premium as fraction of the price, as a function of the deferral period.

Figure 6.4.1: **Price of deferred annuities**



The left panel of this figure displays components of the price of a deferred annuity as a function of the deferral period. The solid curve corresponds to the market price of a deferred annuity; the dashed curve to the actuarially fair price, i.e., the expected discounted cash flows; and the dashed-dotted curve to the risk premium for systematic longevity risk. The right panel of this figure displays the risk premium for systematic longevity risk as a percentage of the price of a deferred annuity.

From Figure 6.4.1 we observe that, as expected, both the price of an annuity and the risk premium are decreasing functions of the deferral period. This occurs because a longer deferral period reduces the (expected) number of payments to be made. The decline in the risk premium for systematic longevity risk is small for short deferral periods, and large for long deferral periods.[17] This occurs because systematic longevity risk for payments to be made for short durations is much smaller than it is for long durations. Finally, we observe that the risk premium as fraction of the price of an annuity is an increasing function of the deferral period. The

[17]The risk premium for a deferred annuity with a long deferral period is high. This occurs because there is a large amount of uncertainty in the probability of surviving until advantaged ages conditional on being alive at the age of 65. In addition, the risk premium is high due to the skewness of the distribution of the probability of surviving.

uncertainty in the survival probabilities is greater for survival probabilities farther in the future, which leads to (relatively) higher risk premiums for payments which have a longer maturity.

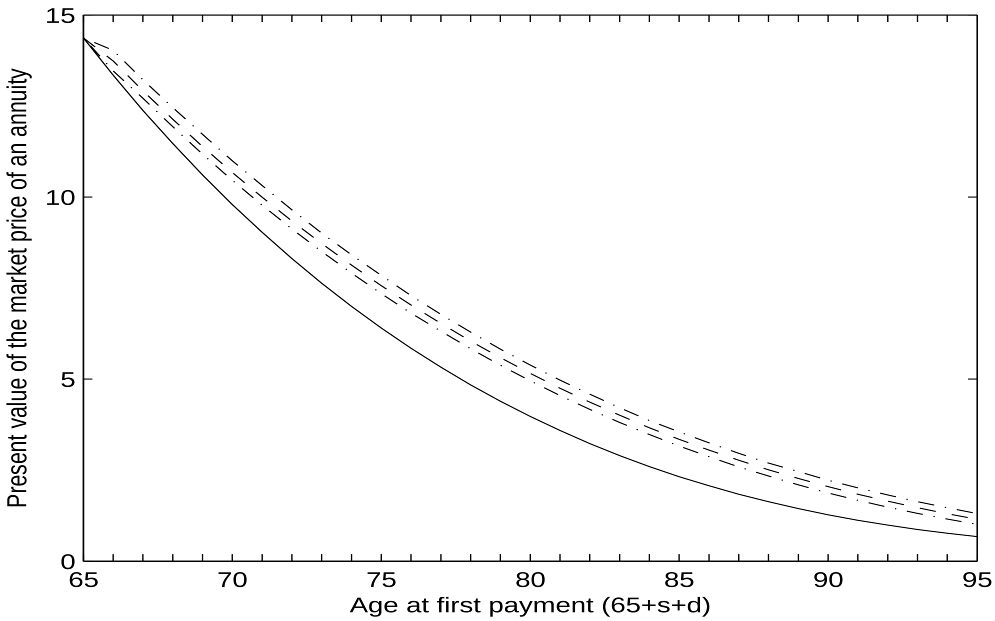
Instead of purchasing a deferred annuity the individual can also postpone the purchase of an immediate annuity. The income stream of a deferred annuity with deferral period d can be mimicked by the following strategy: when the individual is alive at time $d-1$ he purchases an immediate annuity (with $d=1$), and when the individual is not alive at time $d-1$ he does not purchase any annuities. Although the income stream of a deferred annuity can be mimicked using an immediate annuity, when currently purchasing a deferred annuity the price is known, whereas the future price for an immediate annuity is currently stochastic. As new mortality information becomes available, it can be consulted before setting the annuity prices. Therefore, when the annuity purchase is postponed there is generally less uncertainty about the development of future survival probabilities. This reduces the risk premium for longevity risk for an annuity, thus making it more attractive. However, a disadvantage of postponing the purchase of the annuity is that there is currently uncertainty about what the future price of an annuity will be.

Figure 6.4.2 displays the median (dashed curve) and 95% confidence intervals (dashed-dotted curves) of the discounted price of an immediate annuity at date s, i.e., $V_s(A^{(1)}_{s,65+s}) \cdot \left(\frac{1}{1+r^{rf}}\right)^s$, as a function of the postponement period s. For comparison, the current price of a deferred annuity with a nominal yearly payment of one, as a function of the deferral period (solid curve) is also displayed in Figure 6.4.2. From Figure 6.4.2 we observe that it is generally cheaper to currently purchase a deferred annuity than to postpone the purchase of an immediate annuity. Currently purchasing a deferred annuity instead of postponing the purchase of an immediate annuity has the advantage that some of the buyers will not survive until the payoff phase. This can be observed from the following relation between the deferred annuity price and immediate annuity price:

$$V_t\left(A^{(d)}_{x+t,t}\right) = \mathbb{E}^{\mathbb{Q}}_t\left[\left(\frac{1}{1+r^{rf}}\right)^{d-1}\left({}_{d-1}p_{x+t,t}V_{t+d-1}\left(A^{(1)}_{x+d-1,t+d-1}\right) + (1-{}_{d-1}p_{x+t,t})\cdot 0\right)\right].$$

The effect that part of the buyers of a deferred annuity are not alive at time $d-1$ dominates the effect of a risk premium for systematic longevity risk which is generally lower when the moment of purchase is postponed. Note that, even for an individual without any bequest motive, it might still improve utility to postpone the purchase of an annuity instead of currently purchasing a deferred annuity, since the money invested in deferred annuities cannot be invested in equities, which reduces the capital gains from the equity risk premium.

Figure 6.4.2: **Present value of annuity prices**



This figure displays selected quantiles of the present value of the date-s price of an immediate annuity and the price of a deferred annuity as a function of the age at which the initial payment is made. The solid curve corresponds to the current market price of a deferred annuity; the dashed curve corresponds to the present value of the median market price of an immediate annuity purchased at time s; and the dashed-dotted curves correspond to the present value of the 95% confidence bounds of the market price of an immediate annuity purchased at time s.

Let us finally discuss the effect of systematic longevity risk on the attractiveness of annuities. The attractiveness of annuities as an investment opportunity is due to pooling: i.e., the individuals who live longer than expected are subsidized by those who do not. This reallocation of the contributions of those who die to those who survive, is referred to as the mortality credit advantage, see, for example, Milevsky (1998), Milevsky and Young (2002), and Horneff, Maurer, and Stamos (2008). The mortality credit (MC) is defined as the return from currently purchasing an annuity and selling it (at market price) the following year in excess of the risk-free return. This is equivalent to the excess return of purchasing an annuity instead of postponing its purchase to the succeeding year. In a setting without systematic longevity risk the mortality credit is defined as:

$$
\begin{aligned}
MC\left(t,d,x\right) =& \frac{V_{t+1}\left(A_{t+1,x+t+1}^{(\max\{0,d-1\})}\right)+1_{d=0}}{V_t(A_{t,x+t}^{(d)})} - \left(1+r^{rf}\right) && (6.16)\\
=& \frac{1+r^{rf}}{{}_1p_{x+t,t}} - \left(1+r^{rf}\right), && (6.17)
\end{aligned}
$$

where $1_{d=0}$ is an indicator function which equals one if $d=0$, and zero otherwise, and ${}_1p_{x+t,t}$ is the deterministic one-year survival probability. Because ${}_1p_{x+t,t}$ is between zero and one, the mortality credit is always positive. This occurs due to the risk-sharing principle, i.e., in the following year the annuitized wealth is re-allocated to the survivors.

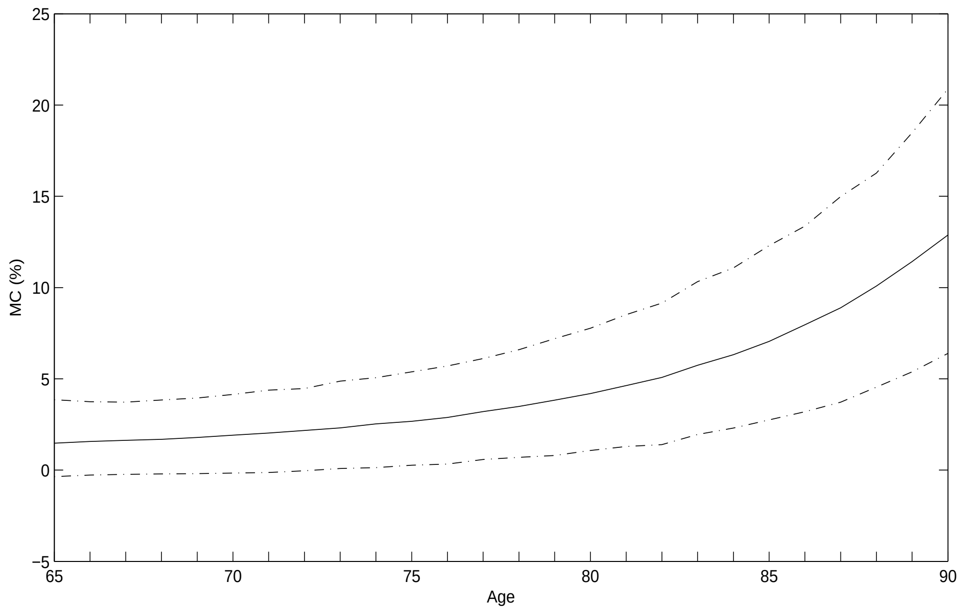
In a setting with systematic longevity risk the mortality credit from (6.16) equals:

$$
MC\left(t,d,x\right) = \frac{\sum_{s\geq\max\{d-1,0\}} \mathbb{E}_{t+1}^{\mathcal{Q}}\left[{}_sp_{x+t+1,t+1}\right]\cdot\left(1+r^{rf}\right)^{-s}+1_{d=0}}{\sum_{s\geq d}\mathbb{E}_t^{\mathcal{Q}}\left[{}_sp_{x+t,t}\right]\cdot\left(1+r^{rf}\right)^{-s}} - \left(1+r^{rf}\right). \qquad (6.18)
$$

Compared to the mortality credit in a setting with deterministic mortality probabilities, the mortality credit in a setting with stochastic mortality probabilities differs in two ways. First, from equation (6.18) we observe that the mortality credit is stochastic instead of deterministic, because $V_{t+1}\left(A_{t+1,x+1}^{(\max\{0,d-1\})}\right)$ depends on the evolution of the mortality probabilities until time $t+1$. Second, the mortality credit is dependent on the deferral period d in a setting with systematic longevity risk, whereas it is independent of the deferral period in a setting without systematic longevity risk. This is due to the fact that the price of an annuity in the following year depends on the change in the distribution of future survival probabilities due to revealed mortality information between time t and time $t+1$. This change may be different for different ages, resulting in various changes to the risk-adjusted expected discounted cash flows of the different annuity payments.

Figure 6.4.3 displays selected quantiles of the distribution of the mortality credit as a function of the age of the individual, for immediate annuities (i.e., $d=1$).

Figure 6.4.3: **Mortality credit for immediate annuities**



This figure displays selected quantiles of the distribution of the mortality credit for immediate annuities under the real-world measure, as a function of the age of the individual. The solid curve corresponds to the median mortality credit for immediate annuities under the real-world measure and the dashed curves correspond to the 95% confidence bounds of the mortality credit for immediate annuities under the real-world measure.

From Figure 6.4.3 we observe that in a setting with systematic longevity risk, in contrast to a setting without systematic longevity risk, the mortality credit can be negative. A negative mortality credit implies that it is cheaper for the individual to currently invest in the risk-free asset and purchase an immediate annuity in the following year than to currently purchase an immediate annuity. This can occur due to systematic longevity risk, which might lead to a change in the distribution of future survival probabilities when new mortality information is revealed. Note that from age 72, the effect of the positive mortality probability is generally larger than the effect of the new information on mortality probabilities, leading to a positive mortality credit with a probability of more than 97.5%. Moreover, recall that the expected excess return of equity is set at 4%, which is lower than the median of the mortality credit for the ages above 80. As discussed in Milevsky and Young

(2002) in a setting without systematic longevity risk, when the mortality credit is higher than the equity risk premium it is optimal for an individual to annuitize all his wealth, because annuities yields a higher expected return with less uncertainty.

6.5 Optimal life-cycle choices

In this section we quantify the effect of the choice of the deferral period and the time at which an annuity is purchased on an individual's expected lifetime utility. Both the price and the payoff stream of an annuity influence the expected lifetime utility. Recall from Section 6.2.1 that the individual is a rational expected lifetime utility maximizer with a CRRA utility function. The individual's preference parameters of the CRRA utility are set equal to values used in the life-cycle literature (see, for example, Gomes and Michaelides, 2005): $\gamma = 5$ and $\beta = 0.96$. The individual is a male currently aged 65, who faces longevity risk and, when he invests in equities he faces investment risk. We assume that the individual invests only once in an annuity and only in one type, i.e., either an immediate annuity ($d = 1$, with $s \geq 0$), or a deferred annuity with a fixed deferral period d (with $s = 0$).

We quantify the attractiveness of the different types of annuities by the certainty equivalent consumption. The certainty equivalent consumption is the yearly consumption level CEC for which the utility of this consumption stream equals the utility given the optimal choices conditional on the type of annuity, $J_0\left(W_0, 0, 0, X_0\right)$. Hence, the certainty equivalent consumption is determined by the following equation:

$$\mathbb{E}_0\left[\sum_{\tau \geq 0} \prod_{s=0}^{\tau-1} p_{x+s,s} \cdot \beta^{\tau} \cdot \frac{(CEC)^{1-\gamma}}{1-\gamma}\right] = J_0\left(W_0, 0, 0, X_0\right), \tag{6.19}$$

with $J_0\left(W_0, 0, 0\right)$ as defined in (6.2). In our results we compare the utility (quantified by the certainty equivalent consumption) obtained by the optimal consumption and investment choices to the utility of the constant consumption level that arises from currently investing all after-consumption wealth in annuities. We refer to this investment strategy as the *fully annuitized (fa) strategy.* The corresponding certainty equivalent consumption in this strategy equals $CEC_{fa} = W_0 / \left(1 + V_0\left(A_0^{(1)}\right)\right)$, where the denominator equals the "price" of a yearly consumption of one, i.e., the current consumption plus the price of an immediate annuity.

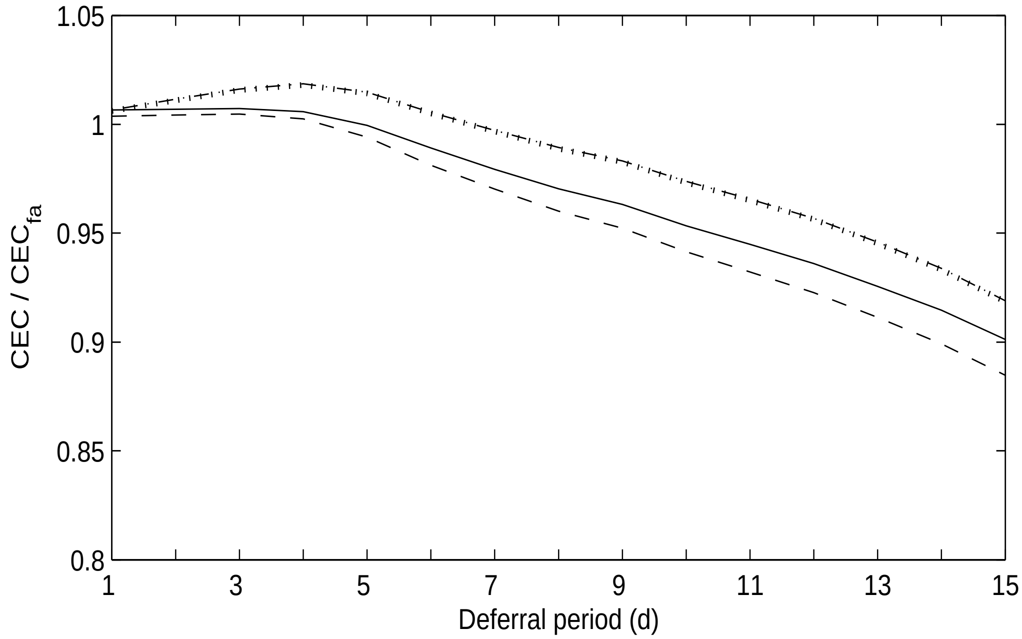
In Section 6.5.1 we investigate the optimal fraction of wealth invested in a deferred annuity and the corresponding certainty equivalent consumption conditional on the immediate purchase of a deferred annuity, i.e., $s = 0$, for different deferral periods. In Section 6.5.2 we investigate the optimal fraction of wealth invested in an immediate annuity and the corresponding certainty equivalent consumption conditional on postponing the purchase of an immediate annuity, i.e., $d = 1$. We assume that the individual purchases an immediate annuity only once.

6.5.1 Purchasing a deferred annuity at retirement date

In this section we investigate the effect of the deferral period on the expected lifetime utility of an individual, conditional on currently ($s = 0$) purchasing a deferred annuity. We maximize the individual's expected lifetime utility as given in (6.2) given constraints (6.3)–(6.6) for annuities with $d = 1, \ldots, 15$ (i.e., different deferral periods, including an immediate annuity), respectively and $s = 0$ (i.e., immediately purchasing an annuity).

First, let us investigate the effect of the deferral period on an individual's expected lifetime utility. Figure 6.5.1 displays the certainty equivalent consumption relative to the certainty equivalent consumption in the fully annuitized strategy, as a function of the deferral period. The figure also illustrates the effect of the pricing method of annuities (i.e., using a risk-neutral approach or using a constant loading factor of 7.3%) on the optimal decision.

Figure 6.5.1: **Certainty equivalent consumption conditional on purchasing a deferred annuity at retirement date**



This figure displays the certainty equivalent consumption relative to the fully annuitized strategy as a function of the deferral period of the annuities. The solid curve corresponds to the certainty equivalent consumption when annuities are priced using risk-neutral pricing with $\lambda = [0.175\ 0.175]'$. The dashed curve corresponds to the certainty equivalent consumption when annuities are priced using the stochastic λ. The dashed-dotted curve corresponds to the certainty equivalent consumption when annuities are priced using a loading factor of 7.3%, irrespective of the deferral period. The dotted curve corresponds to a setting without systematic longevity risk with a loading factor of 7.3%.

In Figure 6.5.1 we observe the following:

i) the optimal annuity is a deferred annuity with a short deferral period;

ii) the effect of systematic longevity risk on the utility gain (or loss) of a longer deferral period is negligibly small.

As expected, we observe that when currently purchasing an immediate annuity (i.e., $d = 1$) the certainty equivalent consumption relative to the fully annuitized strategy is greater than one. This indicates that the investment and consumption choices in the fully annuitized strategy are not the optimal ones. The utility gain, relative to the fully annuitized strategy, is similar in a setting with and without systematic longevity risk, using a constant loading factor for annuity prices. When the annuities are priced using the risk-neutral survival probabilities, the utility gain obtained by purchasing an annuity with the optimal deferral period instead of purchasing an immediate annuity is negligibly small. We also observe that the optimal deferral period is short: only three years when risk-neutral pricing is used (for both $\lambda = [0.175\ 0.175]'$ and the stochastic λ) and four years when a loading factor of 7.3% is used. An increase in the deferral period has two effects, namely:

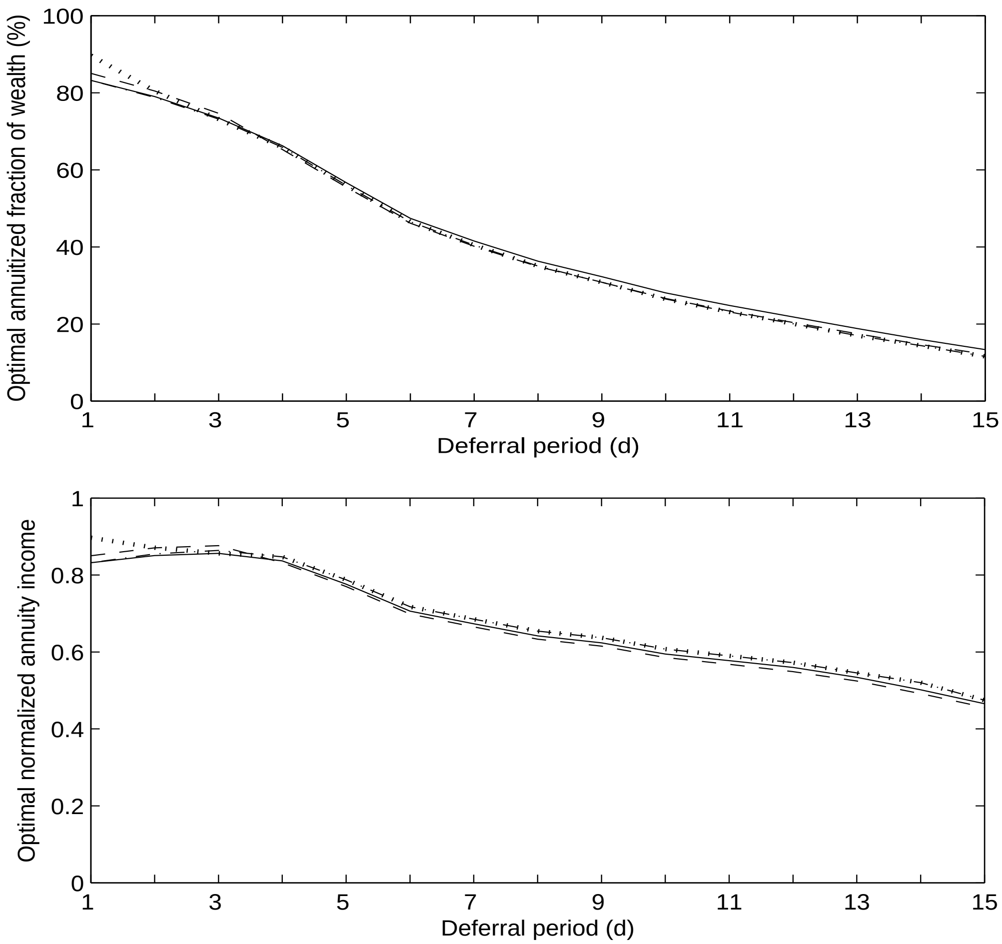
i) A longer deferral period leads to cheaper annuities. Because annuities with a longer deferral period are cheaper the individual can invest more in other assets, i.e., in the risk-free asset or in equities, and/or purchase an annuity with a higher income stream. When the individual chooses to invest a lower proportion of initial wealth in an annuity, a higher fraction of initial wealth may be invested in equities, leading to a higher equity risk premium. When the individual does not choose to invest less in a deferred annuity, a higher deferral period leads to a higher income level in the payoff phase.

ii) A longer deferral period leads to fewer periods with an income guarantee. Hence, there are more periods with greater uncertainty in the consumption level. This uncertainty reduces an individual's expected lifetime utility.

We observe that for deferred annuities with a short deferral period, the first effect dominates. However, for longer deferral periods, the second effect dominates, resulting in a short optimal deferral period.

The attractiveness of an annuity with a longer deferral period is that it is cheaper. As mentioned previously, the reduction in the price of annuities can be used to either increase the income level after the deferral period and/or to increase the fraction of wealth invested in equities or in the risk-free asset. To illustrate this, the upper panel of Figure 6.5.2 displays the optimal fraction of initial wealth which is invested in a deferred annuity as a function of the deferral period. The lower panel of Figure 6.5.2 displays the optimal level of normalized (i.e., relative to the fully annuitized strategy) yearly annuity income after the deferral period as a function of the fixed deferral period.

Figure 6.5.2: **Optimal annuity decision**



This figure displays the optimal annuity decision as a function of the deferral period of the currently purchased annuity. The upper panel displays the optimal fraction of after-consumption wealth which is currently used for the purchase of deferred annuities, i.e., $a_0(d)$. The lower panel displays the optimal annuity income relative to the optimal annuity income in the fully annuitized strategy, i.e., $a_0(d) \cdot V_0\left(A_{65,0}^{(1)}\right)/V_0\left(A_{65,0}^{(d)}\right)$. The solid curve corresponds to annuities priced using the risk-neutral pricing for systematic longevity risk with $\lambda = [0.175\ 0.175]'$. The dashed curve corresponds to annuities priced using the stochastic λ. The dashed-dotted curve corresponds to annuities priced using a loading factor of 7.3%, irrespective of the deferral period. The dotted curve corresponds to a setting without systematic longevity risk and annuities priced using a loading factor of 7.3%, irrespective of the deferral period.

In Figure 6.5.2 we observe the following:

i) the optimal fraction of annuitized wealth is decreasing in the deferral period of the annuity;

ii) after a short deferral period the optimal annuity payment in the payout phase is decreasing in the deferral period;

iii) systematic longevity risk reduces the attractiveness of annuities.

In Figure 6.5.2 we observe that the fraction of wealth which is invested in a deferred annuity is a decreasing function of the deferral period. However, the annuity income level is an increasing function of the deferral period up to a deferral period of three years. This implies that an increase in the deferral period (for $d \leq 3$) initially leads to less annuitized wealth, but in the payoff phase to more wealth invested in annuities. This leads initially to fewer, but in the payoff phase to more capital gains from the mortality credit. This occurs because initially the optimal annuity income increases in the deferral period. When the deferral period is longer than three years both the optimal fraction of annuitized wealth and the optimal annuity payment level are decreasing functions in the deferral period. This occurs because annuities are illiquid. The illiquidity of an annuity restricts the individual's consumption smoothing. Therefore, when the individual is faced with much lower than expected returns in the financial market, he cannot adjust his annuitized wealth level. This may lead to a substantial reduction in the consumption level until the payoff phase of the deferred annuity; this is not optimal.

Let us finally investigate the effect of systematic longevity risk on the optimal fraction of annuitized wealth. Systematic longevity risk may have two effects when the individual currently purchases a deferred annuity. First, it leads to a risk premium which is an increasing function of the deferral period. Compared to a loading factor which is independent of the deferral period, an increasing loading factor leads to a higher fraction of annuitized wealth. In addition, it makes a longer deferral period less attractive. Second, systematic longevity risk leads to stochastic values of the survival probabilities. The uncertainty in the future survival probabilities makes annuities less attractive. In Figure 6.5.2 we observe that excluding systematic longevity risk leads to an increase the optimal the fraction of annuitized wealth when purchasing an immediate annuity from 83.2% to 89.7%.[18] This occurs

[18]For the sake of comparison, we kept the same prices of deferred annuities, using a loading factor of 7.3%.

because systematic longevity risk leads to uncertainty in the utility gain of an annuity; i.e., when future survival probabilities are lower than expected they are less attractive. In the setting with systematic longevity risk the individual adjusts his consumption level to the newly revealed mortality information: he consumes more when the survival probabilities are lower than expected and he saves more when the survival probabilities are higher than expected. In order to be able to adjust his consumption, the individual needs liquid wealth. Although there is a small effect of systematic longevity risk on the fraction of annuitized wealth, the effect of systematic longevity risk on the utility is negligible small. This implies that the utility gain from setting the choices optimal conditional on the newly revealed mortality information is negligible small when the individual purchases deferred annuities. This occurs due to the annuity income, when an individual lives much longer than expected he still has a substantial annuity income at high ages and this reduces the effect of an adjustment in the consumption when new mortality data reveals. Interestingly, the effect of systematic longevity risk on the fraction of annuitized wealth is negligible small after a short deferral period. This occurs because the adjustments of the choices on a change in the distribution of the future survival probabilities are small. In addition, the individual has more liquid wealth and thus is more able to adjust his consumption level as survival probabilities are realized. When the deferral period is larger than three years the difference in the optimal fraction of annuitized wealth with and without systematic longevity risk is negligibly small, due to the substantial amount of liquid wealth. Hence, the exclusion of systematic longevity risk leads to a higher utility for an immediate annuity, but not for a deferred annuity (with a moderate or long deferral period).

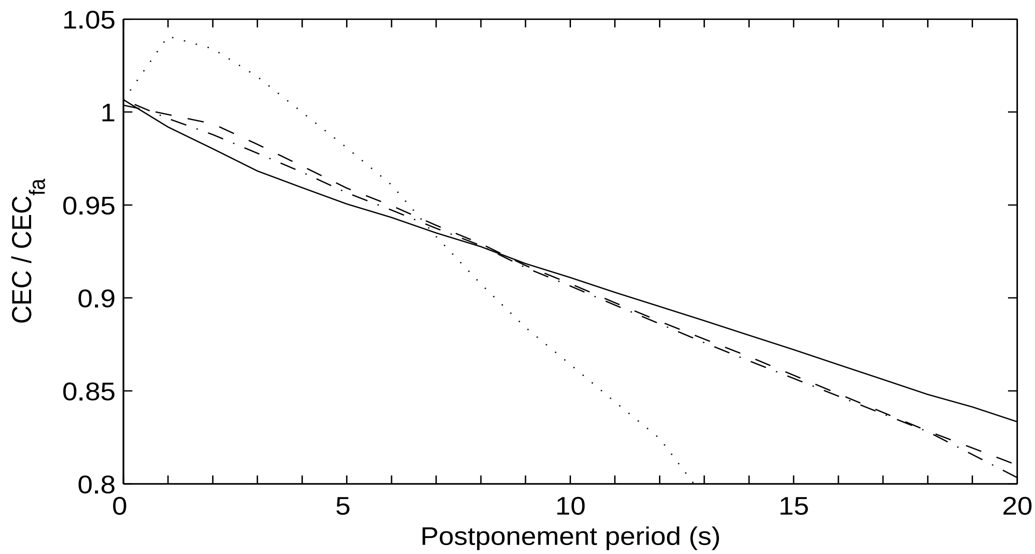
Let us now discuss how our findings relate to the existing literature on deferred annuities. For our representative agent it would be optimal to purchase a deferred annuity which starts with an initial payout at the age of 68. Our finding deviates from the literature that argues that deferred annuities with a long deferral period are preferable (see Milevsky, 2005; Dus, Maurer, and Mitchell, 2005; Horneff and Maurer, 2008; and Gong and Webb, 2009). Milevsky (2005) show that, under the assumption of a real interest rate of 3.25%, that deferred annuities have a higher money's worth. Dus, Maurer, and Mitchell (2005) show that deferred annuitization with a fixed withdrawal rule can enhance expected payouts and cut expected shortfall risk. Horneff and Maurer (2008) show, in a life cycle model with labor income risk and equity risk, that during the working life buying deferred annuities where payouts start either at age 65 or at age 85 and may increase the expected lifetime utility relative to buying immediate annuities during the working phase. Gong and Webb (2009) show that for a couple, with only non-systematic longevity risk, when the risk aversion is low and the annuitant money's worth is low enough, using a naive decumulation strategy (the rule-of-thumb strategy is an equal consumption in all periods prior to the age at which the ALDA payments commence) it may be optimal to purchase a deferred annuity with payments commencing at the age of 85. Our findings primarily differ from existing literature since we determine the optimal choices dynamically using the objective function to maximize an individual's expected lifetime utility. This differs from existing literature which uses either expected shortfall risk using rules of thumbs for the choices or using an individual's utility where an individual's choices are determined using rules of thumbs. The existing literature argues that a deferred annuity allows for mortality credits when they are high, and allows for liquid wealth to earn the equity risk premium when they are low. We find that the low mortality credit just after the annuity transaction does not offset the utility loss because of the illiquidity of the annuity. The illiquidity of annuities restricts an individual's opportunities to smooth consumption. This affects the expected lifetime utility, especially when the individual is faced with adverse shocks in the equity market before the payoff phase. Therefore, conditional on the purchase of a deferred annuity with a long deferral period, it is not optimal to invest a large proportion of the liquid wealth in an annuity; i.e., the individual needs much more precautionary saving when the deferral period is long. This is in line with the results of Bayraktar and Young (2009). They found that immediate annuities are more preferable than deferred annuities, in a setting where an individual minimizes lifetime ruin probability instead of maximize expected lifetime utility.

6.5.2 Postponing the purchase of an immediate annuity

An alternative to purchasing a deferred annuity at retirement date is to postpone the purchase of an immediate annuity. The advantage of postponing the annuity purchase is that the individual maintains only liquid assets until the moment the annuity is purchased. This allows the individual to adjust his consumption and annuity income level to the realizations of the financial market, at the moment the annuity is purchased. The disadvantage of postponing the annuity purchase is that the price of the annuity is currently stochastic and that the individual does not receive the mortality credit until the moment of the purchase of an annuity. In this section we maximize the individual's expected lifetime utility as given in (6.2) given constraints (6.3)–(6.6) for an immediate annuity (i.e., $d = 1$) with $s = 0, \ldots, 20$ (i.e., different fixed postponement periods, with $s = 0$ immediately purchasing an annuity).

Figure 6.5.3 displays the certainty equivalent consumption conditional on purchasing an immediate annuity at time s, relative to the currently fully annuitized strategy, and as a function of the postponement periods. The individual maximizes his utility by investing in the risk-free asset and in equities, and, at time s, in an annuity.

Figure 6.5.3: **Certainty equivalent consumption conditional on the postponement period of an immediate annuity**



This figure displays the certainty equivalent consumption relative to the fully annuitized strategy as a function of the postponement period of the purchase of immediate annuities. The solid curve corresponds to the certainty equivalent consumption when annuities are priced using risk-neutral pricing with $\lambda = [0.175\ 0.175]'$. The dashed curve corresponds to the certainty equivalent consumption when annuities are priced using the stochastic λ. The dashed-dotted curve corresponds to the certainty equivalent consumption when annuities are priced using a loading factor of 7.3%, irrespective of the deferral period. The dotted curve corresponds to the certainty equivalent consumption in a setting without systematic longevity risk and annuities are priced using a loading factor of 7.3%, irrespective of the deferral period.

Let us first investigate the effect of postponing the purchase of an immediate annuity on an individual's expected lifetime utility. In Figure 6.5.3 we observe that with systematic longevity risk the certainty equivalent consumption is decreasing in the postponement period. The three main effects of postponing the annuity purchase on the expected lifetime utility of the individual are:

i) *Mortality credit.* By postponing the annuity decision the individual does not earn the mortality credit of the annuity until the moment of purchasing the annuity.

ii) *Equity risk premium.* By postponing the annuity decision the individual has until time s only liquid wealth. Therefore, he is less restricted in the fraction of total wealth he invests in equities. This might lead to a higher capital gain from the equity risk premium.

iii) *Conversion rate risk.* Due to stochastic survival probabilities the price of an annuity in the future is stochastic, which results in conversion rate risk.

We observe that the positive effect of postponing the annuity purchase, i.e., the equity risk premium, is smaller than the negative effects, i.e., the missed mortality credit and the conversion rate risk. This occurs because just after retirement, the missed mortality credit is small, but the conversion rate risk is substantial. This conversion rate risk is affected by changes in survival probabilities in two ways. The annuity prices in the future are affected by the difference in realized and expected survival probabilities on the one hand and by a change in the expected trend in the evolution of future survival probabilities on the other hand.

Next, let us investigate the effect of systematic longevity risk on the effect of an individual's expected lifetime utility conditional on postponing the purchase of an immediate annuity. In Figure 6.5.3 we observe that the inclusion of systematic longevity risk has two effects, namely:

i) whereas it is optimal to postpone the purchase of an immediate annuity in a setting without systematic longevity risk, this is not the case in a setting with systematic longevity risk;

ii) after a short postponement period the utility loss of increasing the postponement period with an additional year is much larger in a setting without longevity risk than in a setting with systematic longevity risk.

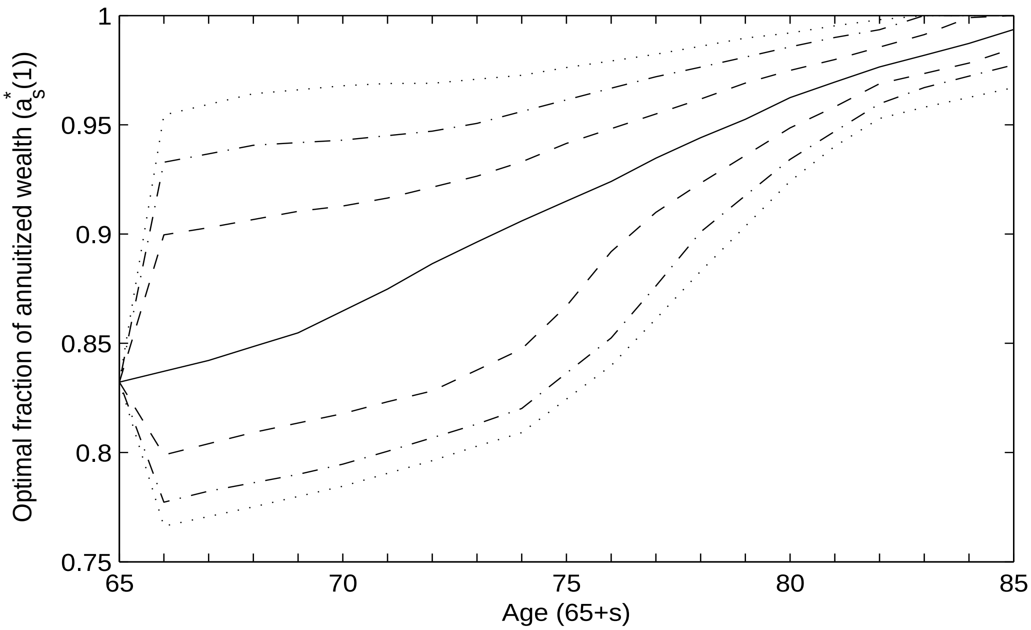
There are two opposite effects of systematic longevity risk on the expected lifetime utility when the purchase of an annuity is postponed. On the one hand systematic longevity risk reduces the attractiveness of an immediate annuity, as discussed in Section 6.5.1. On the other hand, systematic longevity risk leads to uncertainty in the future prices of annuities which leads to a lower utility when the annuity purchase is postponed in a setting with systematic longevity risk than in one without it. When the postponement period is short, excluding systematic longevity risk and thus a deterministic price of annuities purchased in the future leads to a higher utility when the annuity purchase is postponed. When the postponement period is longer, excluding systematic longevity risk leads to a larger decrease in the certainty equivalent consumption. This is due to the fact that immediate annuities are more attractive in a setting without systematic longevity risk than in a setting with systematic longevity risk.

Figure 6.5.4 displays selected quantiles of the optimal fraction of annuitized wealth as a function of the postponement period (i.e., as function of s). Note that the individual has CRRA preferences, which implies that the optimal fraction of annuitized wealth is independent of the past equity returns and the wealth level of the individual.[19] This implies that the optimal fraction of wealth annuitized is only affected by the uncertainty in the survival probabilities. First, in Figure 6.5.4 we observe that, as expected, the fraction of wealth invested in an immediate annuity generally increases with the length of the postponement period, or equivalently, with the age of the individual. This occurs because the mortality probabilities increase with age which generally leads to a higher mortality credit (see Figure 6.4.3).

Second, in Figure 6.5.4 we also observe that the uncertainty in the distribution of the optimal fraction of annuitized wealth is large. Therefore, we investigate how the evolution of future survival probabilities affects the fraction of annuitized wealth. To do so, we decompose the total effect in an effect due to changes in the risk premium for systematic longevity risk (which is primarily affected by changes in the uncertainty in the future survival probabilities) and an effect due to changes in the actuarial fair value (which is primarily affected by changes in expected survival probabilities). We calculate the correlation between the optimal fraction of wealth invested in an annuity and the actuarially fair value. We find that they are positively

[19]Note that this only holds when the return dynamics of the asset portfolio are modeled in such a way that past returns do not have an influence on current and future returns. In our setting the return process is stationary, since we use a random walk with drift process to model the price of equities.

Figure 6.5.4: **The optimal fraction of wealth invested in an annuity conditional on the postponement period**



This figure displays the selected quantiles of the optimal fraction of after-consumption wealth which is invested in annuities as a function of the age of the individual at the moment of purchase of the immediate annuity. The solid curve corresponds to the median; the dashed curves correspond to the 50% confidence bounds; the dashed-dotted curves correspond to the 80% confidence bounds; and the dotted curves correspond to the 90% confidence bounds.

correlated.[20] This occurs because a change in the expected survival probabilities has a small effect on the attractiveness of an annuity, but a much larger effect on the price of an annuity. We also calculate the correlation between the optimal fraction of wealth invested in an annuity and the risk premium for systematic longevity risk in the annuity, and we find a negative correlation.[21] This occurs because an increase in the risk premium in an annuity leads to a higher cost without additional expected payments, making an annuity less favorable. Interestingly, the effect of a change in the actuarially fair value on the optimal fraction of annuitized wealth is larger

[20]The correlation between the actuarially fair value and the fraction of annuitized wealth is approximately 0.8 depending on the time of the purchase of an annuity.

[21]The correlation between the risk margin and the fraction of annuitized wealth is between 0 and -0.3, and a decreasing function in the time of the purchase of an annuity.

than the effect of a change in the risk premium. This occurs because a change in the actuarially fair value is due to a change in the expected survival probabilities. The consequence of higher survival probabilities is a higher actuarially fair value of an annuity on the one hand, and a larger consumption effect at higher ages on the expected lifetime utility on the other hand. Therefore, when the individual expects to live longer due to newly revealed survival information, the optimal fraction of annuitized wealth increases. This is due to the higher price of an annuity and due to a higher weight in the expected lifetime utility function of consumption at advanced ages.

6.6 Alternative individual characteristics and financial market parameters

The results shown in the previous sections suggest that systematic longevity risk may significantly affect an individual's annuity decision. In this section we show that these results are robust to alternative assumptions in individual characteristics and financial market parameters. In the existing literature (see, for example, Blake, Cairns, and Dowd, 2003; Horneff, Maurer, and Samos, 2008; and Babel and Merrill, 2007) different assumptions on the risk aversion coefficient and the equity risk premium are used. In this section we investigate the robustness of the results for changes in the risk aversion coefficient and in the equity risk premium. In particular, we compute the optimal annuity decisions for a less risk averse individual, i.e., an individual with a risk aversion coefficient of 2 instead of 5. Moreover, we compute the optimal annuity decisions when the expected excess return on equities is 7% (i.e., $\lambda^s = 0.343$) instead of 4% (i.e., $\lambda^s = 0.155$). As discussed in Section 6.3.1, in the standard life-cycle model literature the excess return is set lower than the empirical one in order to cope with transaction costs.

First, we investigate the robustness of the utility-loss when postponing the annuity purchase. We find that for the different alternative individual characteristics and financial market parameters, i.e., for both a lower value of the risk aversion parameter ($\gamma = 2$, with $\lambda^s = 0.155$) and for a higher equity risk premium ($\lambda^s = 0.343$, with $\gamma = 5$) it is optimal not to postpone the annuity purchase. Either a lower risk aversion ($\gamma = 2$) or a higher equity risk premium ($\lambda^s = 0.343$) does lead to a smaller utility loss when the annuity purchase is postponed than in the case where $\gamma = 5$ and $\lambda^s = 0.155$. In a setting with systematic longevity risk, for the investigated values of the risk aversion and the equity risk premium, we find that it is only optimal to postpone the annuity purchase when both the risk aversion is low and the equity risk premium is high ($\gamma = 2$ and $\lambda^s = 0.343$). Assuming a higher equity risk premium ($\lambda^s = 0.343$), we find that the optimal fraction of wealth currently invested in an immediate annuity is approximately 60%. This is lower than the average observed fraction of annuitized wealth of the current retiring US cohort, taking into account the pre-existing annuitized wealth, such as social security benefits and defined benefit pension plans (see Dushi and Webb, 2004a, using data of the Health and Retirement Survey). Although a high equity risk premium of 7% explains the empirical level of annuitization this equity risk premium seems empirically to be too high (see the discussion in Section 6.3.1). One possible explanation of the annuity puzzle might be that the individual does not take transaction costs into account. Hence, when the individual would not be rational and expects a too high equity risk premium, this might explain the individual's choice to forgo annuitization.

Next, we investigate the robustness of the optimal deferral period to alternative individual characteristics and financial market parameters. Table 6.6.1 displays the optimal deferral period under the different assumptions.

Table 6.6.1: **Optimal deferral period**

γ	λ^s	pricing annuities	$d^\star$	gain in CEC
5	0.155	$\lambda = [0.175\, 0.175]'$	3	0.06%
5	0.155	stochastic λ	3	0.10%
5	0.155	loading factor (7.3%)	4	1.19%
5	0.343	$\lambda = [0.175\, 0.175]'$	3	0.07%
2	0.155	$\lambda = [0.175\, 0.175]'$	5	0.30%

This table displays the optimal deferral period of annuities currently purchased for several alternatives of the parameters in the model. The first column corresponds to the risk aversion parameter; the second column corresponds to the parameter for the expected excess return on equity (i.e., $\lambda^s = 0.155$ if the expected excess return is 4% and $\lambda^s = 0.343$ if the expected excess return is 7%); the third column corresponds to the pricing of annuities (using a loading factor of 7.3% or using risk-neutral survival probabilities); the fourth column corresponds to the optimal deferral period; and the last column corresponds to the gain in certainty equivalent consumption when purchasing an annuity with the optimal deferral period instead of purchasing an immediate annuity.

In this table we observe that the optimal deferral time is short, also in cases where risk aversion is lower or the equity risk premium is higher. As expected, when risk aversion is low enough or the return on an alternative to an annuity payment (i.e., the expected return on equity) is higher, it becomes more favorable to defer the first annuity payment. When deferring the first annuity payment, the individual optimally invests a lower fraction of wealth in a deferred annuity at retirement date, which is partly invested in equities, depending on risk aversion. The utility gain is very small when purchasing an annuity with the optimal deferral period instead of an immediate one and this result is robust for all five examined alternatives of the parameters of the individual characteristics and the financial market.

6.7 Conclusions

This chapter investigates the effect of systematic longevity risk on an individual's optimal annuitization decision in a life-cycle model. In addition, we investigate the optimal annuity product an individual should purchase, i.e., a deferred annuity at retirement date or an immediate annuity purchased either at retirement date or at a fixed time in the future. We argue that systematic longevity risk affects the optimal annuity decision in three ways. First, due to systematic longevity risk the price of an annuity purchased in the future depends on the distribution of future survival probabilities, and therefore it is currently stochastic. Second, systematic longevity risk affects the current market price of a deferred annuity. The impact of systematic longevity risk on the market price of an annuity depends on the type of the annuity. For payments with greater longevity uncertainty (payments at advanced ages), the risk premium as fraction of the expected discounted cash flow of the payment, is much higher than for payments with smaller longevity uncertainty (payments in the first years following the purchase of an annuity). Compared to a loading factor which is independent of the deferral period, this makes deferred annuities less attractive. Third, systematic longevity risk leads to annuities which are less attractive. This occurs because the systematic longevity risk leads to uncertainty in the value of annuities, thereby making annuities a more risky investment.

In the context of our life-cycle model we show that systematic longevity risk affects an individual's annuity decision in two ways. First, due to the uncertainty in the future prices of annuities, it is utility-increasing for an individual aged 65 to purchase an annuity currently instead of postponing the annuity purchase. This differs from the existing literature, which ignores systematic longevity risk. Second, we show that systematic longevity risk makes deferred annuities less attractive. We find that it is optimal to currently purchase an annuity with a short deferral period. However, we find that the utility loss of currently purchasing an immediate annuity, instead of currently purchasing an annuity with the optimal deferral period, is negligibly small.

This chapter can be extended in several ways. For example, many modifications and extensions can be made with respect to the CRRA utility function, such as adding a bequest motive. Obviously, this will affect the quantitative results. However, since the mechanisms as described in this chapter will probably remain in place, the qualitative results, i.e., the effect of systematic longevity risk and the effect of an increase in the deferral period, will probably also remain in place. This chapter could also be extended by including other types of annuities, such as variable annuities, or different options in the annuity (for example, a period-certain, or a lump-sum option), allowing for the purchase of a portfolio of different types of annuities, or allowing the individual to gradually purchase annuities. For example, the model could be extended by allowing for the postponement of the purchase of deferred annuities. However, this might not be optimal, because, as observed in the case of immediate annuities, it is utility-reducing to postpone the annuity purchase due to the currently stochastic price of these annuities. It would be more interesting to extend the literature on optimal gradual annuitization by allowing for a combination of annuity products purchased at different moments in the life-cycle.

This chapter extends the current literature on life cycle models in two ways. First, we allow for systematic longevity risk as a risk factor for the individual. Second, for the literature on literature on life cycle model investigating the optimal deferral period, we include other sources of risk than only non-systematic longevity risk. We allow also for systematic longevity risk and equity risk. Individuals also faces other kinds of risk, such as health risk, housing risk, and interest rate risk. Peijnenburg, Nijman, and Werker (2009) show that including health risk makes annuitizing at retirement age more favorable and this would enforce the result we found. Housing wealth is a risky asset, like equities, but it is also different from equities since it provides a constant utility stream and selling a house will lead to moving costs.

6.A Method to calculate optimal annuity decision

6.B Method to calculate optimal annuity decision

In this appendix we describe the technique to obtain optimal choices in a life-cycle model which was proposed by Brandt, Goyal, Santa-Clara, and Stroud (2005) and by Carroll (2006), with several extensions proposed by Koijen, Nijman, and Werker (2009).

Let us first summarize the individual's life-cycle problem. For the parameter estimates in the CBD model, as given in equations (6.12) and (6.13) we denote

$$\widehat{\mu}_t = [\widehat{\mu}_t^{(1)}\ \widehat{\mu}_t^{(2)}]',\ \ \mu = [\mu^{(1)}\ \mu^{(2)}]',$$

$$\widehat{V}_t = \begin{pmatrix} \widehat{V}_t^{(1,1)} & \widehat{V}_t^{(1,2)} \\ \widehat{V}_t^{(1,2)} & \widehat{V}_t^{(2,2)} \end{pmatrix}, \widehat{C}_t = \begin{pmatrix} \widehat{C}_t^{(1,1)} & \widehat{C}_t^{(1,2)} \\ 0 & \widehat{C}_t^{(2,2)} \end{pmatrix}, \text{ and } C = \begin{pmatrix} C^{(1,1)} & C^{(1,2)} \\ 0 & C^{(2,2)} \end{pmatrix}.$$

Let $\widehat{\theta}_t = [\widehat{\mu}_t^{(1)}\ \widehat{\mu}_t^{(2)}\ \widehat{V}_t^{(1,1)}\ \widehat{V}_t^{(1,2)}\ \widehat{V}_t^{(2,2)}\ ]'$ be the vector of the maximum likelihood estimates of the parameters of the stochastic survival process based on the information revealed up to time t and let $\mathbb{E}_t\left[\cdot\right]$ be the expectation conditional on the exogenous and endogenous state variables at time t.

Formally, as described in Section 6.2.1, the individual solves:

$$J_0\left(W_0, A_0, B_0, X_0\right) = \max_{a_s(d),\{w_\tau, C_\tau\}_{\tau\geq 0}} \left\{ \mathbb{E}_0\left[\sum_{\tau\geq 0} \prod_{s=0}^{\tau-1} p_{x+s,s} \cdot \beta^\tau \cdot \frac{(C_\tau)^{1-\gamma}}{1-\gamma} \right] \right\}.$$

The individual's optimization problem is subject to liquidity and short-selling constraints, and given the endogenous (A_0, B_0, and W_0) and exogenous (X_0) state variables. The liquidity and short-selling constraints are:

$$\begin{aligned} 0 \leq a_s(d) \leq & 1, & \\ 0 \leq w_\tau \leq & 1, & \text{for } \tau \geq 0, \\ C_\tau \leq & W_\tau + A_\tau, & \text{for } \tau \geq 0. \end{aligned}$$

The evolution of the endogenous state variable W_τ is given by:

$$W_{\tau+1} = \begin{cases} (W_\tau - C_\tau) \cdot (1 - a_s(d)) \cdot \left(1 + r^{rf} + w_\tau \cdot \left(r_\tau - r^{rf}\right)\right), & \text{if } \tau = s, \\ (W_\tau + A_\tau - C_\tau) \cdot \left(1 + r^{rf} + w_\tau \cdot \left(r_\tau - r^{rf}\right)\right), & \text{if } \tau \neq s, \end{cases}$$

The time-t value of the endogenous state variables A_t and B_t are given by:

$$A_{t+1} = \begin{cases} A_t, & \text{if } t \neq s+d-1, \\ B_t, & \text{if } t = s+d-1, d > 1, \\ \frac{a_s(1)\cdot(W_s - C_s)}{V_s\left(A^{(1)}_{x+s,s}\right)}, & \text{if } t = s, d = 1, \end{cases}$$

with $A_0 = 0$, and

$$B_{t+1} = \begin{cases} B_t, & \text{if } t \neq s, \\ \frac{a_s(d)\cdot(W_s - C_s)}{V_s\left(A^{(d)}_{x+s,s}\right)}, & \text{if } t = s, \end{cases}$$

with $B_0 = 0$. The state variable B_t does only play a role when $d > 1$. The time-t values of the probability that an $x + t$-years old individual will survive another τ years, ${}_\tau p_{x+t,t}$, for $\tau > 0$, are given by:

$$\begin{aligned} {}_\tau p_{x+t,t} = \prod_{s=t}^{t+\tau-1} p_{x+t+s,s} &= \prod_{s=0}^{\tau-1} (1 - {}_1q_{x+t+s,t+s}) \\ &= \prod_{s=0}^{\tau-1} \frac{1}{1 + \exp\left(k^{(1)}_{t+s} + (x+t+s)\cdot k^{(2)}_{t+s} + \epsilon_{x+t+s,t+s}\right)}. \end{aligned}$$

Systematic longevity risk affects the survival probabilities generating additional uncertainty in our life-cycle model. As defined in equations (6.10) and (6.11), for the process of the survival probabilities we assume that there exists parameter risk in the distribution of future survival probabilities. We assume that the individual updates the parameters in the CBD model. The distribution of the parameters in the CBD model is given by:

$$\begin{aligned} V^{-1}|D \sim &\text{Wishart}\left(\tau + \bar{t} - \underline{t}, (\tau + \bar{t} - \underline{t} + 1)^{-1}\widehat{V}_\tau^{-1}\right), \\ \mu|V, D \sim &MVN\left(\widehat{\mu}_\tau, (\tau + \bar{t} - \underline{t} + 1)^{-1}V\right), \end{aligned}$$

where $\widehat{\mu}_\tau$ and $\widehat{V}_\tau = \widehat{C}'_\tau \widehat{C}_\tau$ are the maximum likelihood estimates of μ and V at time τ. The evolution of the individual's information on the exogenous state variables is given by:

$$\begin{aligned} \begin{pmatrix} k^{(1)}_{t+1} \\ k^{(2)}_{t+1} \end{pmatrix} &= \begin{pmatrix} k^{(1)}_t \\ k^{(2)}_t \end{pmatrix} + \begin{pmatrix} \widehat{\mu}^{(1)} \\ \widehat{\mu}^{(2)} \end{pmatrix} + \begin{pmatrix} \widehat{C}^{(1,1)} & \widehat{C}^{(1,2)} \\ 0 & \widehat{C}^{(2,2)} \end{pmatrix} \cdot \begin{pmatrix} N^{(1)}_t \\ N^{(2)}_t \end{pmatrix} \\ \widehat{\theta}_{t+1} &= \widehat{\theta}_t + \Delta_{\widehat{\theta}_t}, \end{aligned}$$

with $\Delta_{\widehat{\theta}_t} = f(t, \widehat{\theta}_t, N_t)$, which is equal to:

$$\begin{pmatrix} \frac{\widehat{C}_t^{(1,1)} N_t^{(1)}}{t+1+\bar{t}-\underline{t}} \\ \frac{\widehat{C}_t^{(2,2)} N_t^{(2)}}{t+1+\bar{t}-\underline{t}} \\ \frac{(t+\bar{t}-\underline{t})}{t+1+\bar{t}-\underline{t}} (\widehat{\mu}_t^{(1)})^2 - (\widehat{\mu}_t^{(1)} + \frac{\widehat{C}_t^{(1,1)} N_t^{(1)}}{t+1+\bar{t}-\underline{t}})^2 + \frac{(\widehat{\mu}_t^{(1)} + \widehat{C}_t^{(1,1)} N_t^{(1)})^2 - \widehat{V}_t^{(1,1)}}{t+1+\bar{t}-\underline{t}} \\ \frac{(t+\bar{t}-\underline{t}) \widehat{\mu}_t^{(1)} \widehat{\mu}_t^{(2)}}{t+1+\bar{t}-\underline{t}} - (\widehat{\mu}_t^{(1)} + \frac{\widehat{C}_t^{(1,1)} N_t^{(1)}}{t+1+\bar{t}-\underline{t}})(\widehat{\mu}_t^{(2)} + \frac{\widehat{C}_t^{(2,2)} N_t^{(1)}}{t+1+\bar{t}-\underline{t}}) + \frac{(\widehat{\mu}_t^{(1)} + \widehat{C}_t^{(1,1)} N_t^{(1)})(\widehat{\mu}_t^{(2)} + \widehat{C}_t^{(2,2)} N_t^{(2)}) - \widehat{V}_t^{(1,2)}}{t+1+\bar{t}-\underline{t}} \\ \frac{(t+\bar{t}-\underline{t})}{t+1+\bar{t}-\underline{t}} (\widehat{\mu}_t^{(2)})^2 - (\widehat{\mu}_t^{(2)} + \frac{\widehat{C}_t^{(2,2)} N_t^{(1)}}{t+1+\bar{t}-\underline{t}})^2 + \frac{(\widehat{\mu}_t^{(2)} + \widehat{C}_t^{(2,2)} N_t^{(2)})^2 - \widehat{V}_t^{(2,2)}}{t+1+\bar{t}-\underline{t}} \end{pmatrix}.$$

The individual updated information at time t on the distribution of the survival probabilities is fully captured by $X_t = \left[k_t^{(1)} \; k_t^{(2)} \; \widehat{\theta}_t' \right]'$.

In Appendix 6.B.1 we provide the normalizations which we use later in the appendix. In Appendix 6.B.2 we describe the method to obtain the currently optimal level of annuitized wealth, conditional on the purchase of a deferred annuity with a fixed deferral period. Finally, in Appendix 6.B.3 we describe the method to obtain the optimal level of annuitized wealth, conditional on a fixed postponement period before purchasing an immediate annuity.

6.B.1 Renormalization

In this section we present the normalizations. To improve the readability, in this appendix we drop the first argument in the value function J_t. Let $u(C_t) = \frac{C_t^{1-\gamma}}{1-\gamma}$. The individual solves:

$$J_t\left(W_t, A_t, B_t, X_t\right) = \max_{\{w_t, C_t\}} \left\{ \frac{C_t^{1-\gamma}}{1-\gamma} + \beta \mathbb{E}_t \left[p_{x+t,t} J_{t+1}\left(W_{t+1}, A_{t+1}, B_{t+1}, X_{t+1}\right) \right] \right\}$$

$$\text{s.t.}$$

$$W_{t+1} = \begin{cases} \left(W_t + A_{t+1} - C_t\right)\left(1 + r^{rf} + w_t\left(r_t - r^{rf}\right)\right) & \text{if } t \neq s \\ \left(W_s - C_s\right)\left(1 - a_s\right)\left(1 + r^{rf} + w_s\left(r_s - r^{rf}\right)\right) & \text{if } t = s \end{cases} \tag{6.20}$$

$$A_{t+1} = \begin{cases} A_t & \text{if } t \neq s + d - 1 \\ B_t & \text{if } t = s + d - 1, d > 1 \\ \frac{W_s - C_s}{V_s(A_{x+s,s}^{(d)})} \cdot a_s & \text{if } t = s, d = 1 \end{cases} \tag{6.21}$$

$$C_t \leq W_t + A_t \tag{6.22}$$

$$B_{t+1} = \begin{cases} B_t, & \text{if } t \neq s, \\ \frac{a_s(d) \cdot (W_s - C_s)}{V_s\left(A_{x+s,s}^{(d)}\right)}, & \text{if } t = s. \end{cases} \tag{6.23}$$

We start by considering the time after the decision of annuity income where the individual has a fixed yearly annuity income level, i.e., at time $t > s$. First, we consider this problem at the last period, i.e., at time $MA - x$. The utility of the individual is given by:

$$J_{MA-x}\left(W_{MA-x}, A_{MA-x}, X_{MA-x}\right) = \frac{\left(W_{MA-x} + A_{MA-x}\right)^{1-\gamma}}{1-\gamma}. \tag{6.24}$$

Let the variables with an overline be the variables without an overline, divided by the annuity income,[22] then we can rewrite (6.24) as:

$$\begin{aligned} J_{MA-x}\left(W_{MA-x}, A_{MA-x}, X_{MA-x}\right) =& A_{MA-x}^{1-\gamma} \cdot \frac{\left(\overline{W}_{MA-x} + 1\right)^{1-\gamma}}{1-\gamma} \\ =& A_{MA-x}^{1-\gamma} \cdot J_{MA-x}\left(\overline{W}_{MA-x}, 1, X_{MA-x}\right). \end{aligned} \tag{6.25}$$

Also the wealth equation and budget constraint can be rewritten as:

$$\begin{aligned} W_{t+1}/A_t =& \frac{W_t + A_t - C_t}{A_t} \cdot \left(1 + r^{rf} + w_t \cdot \left(r_t - r^{rf}\right)\right) \Leftrightarrow \\ \overline{W}_{t+1} =& \left(\frac{W_t}{A_t} + \frac{A_t}{A_t} - \frac{C_t}{A_t}\right) \cdot \left(1 + r^{rf} + w_t \cdot \left(r_t - r^{rf}\right)\right) \\ =& \left(\overline{W}_t + 1 - \overline{C}_t\right) \cdot \left(1 + r^{rf} + w_t \cdot \left(r_t - r^{rf}\right)\right) & (6.20') \\ \overline{C}_t \leq& \overline{W}_t + 1, & (6.22') \end{aligned}$$

where (6.20') and (6.22') are (6.20) and (6.22) reformulated in terms of variables with an overline.

Next, consider $t = MA - x - 1$. Using (6.25) we have:

$$\begin{aligned} J_t\left(W_t, A_t, X_t\right) =& \max_{\{w_t, C_t\}} \left\{\frac{C_t^{1-\gamma}}{1-\gamma} + \beta \cdot \mathbb{E}_t\left[p_{x+t,t} \cdot J_{t+1}\left(W_{t+1}, A_{t+1}, X_{t+1}\right)\right]\right\} \\ & \text{s.t. } (6.20)\text{-}(6.22) \\ =& \max_{\left\{w_t, \overline{C}_t\right\}} \left\{A_t^{1-\gamma} \frac{\overline{C}_t^{1-\gamma}}{1-\gamma} + \beta \mathbb{E}_t\left[p_{x+t,t} J_{t+1}\left(\overline{W}_{t+1}, 1, X_{t+1}\right) A_{t+1}^{1-\gamma}\right]\right\} \\ & \text{s.t. } (6.21), (6.20'), (6.22'), \overline{W}_t = W_t/A_t, \text{ and } \overline{C}_t = C_t/A_t \\ =& A_t^{1-\gamma} \cdot \max_{\left\{w_t, \overline{C}_t\right\}} \left\{\frac{\overline{C}_t^{1-\gamma}}{1-\gamma} + \beta \cdot \mathbb{E}_t\left[p_{x+t,t} \cdot J_{t+1}\left(\overline{W}_{t+1}, 1, X_{t+1}\right)\right]\right\} \\ & \text{s.t. } (6.21), (6.20'), (6.22'), \overline{W}_t = W_t/A_t, \text{ and } \overline{C}_t = C_t/A_t. \end{aligned} \tag{6.26}$$

[22] Assuming $A_t > 0$ for $t > s$, i.e., the individual has a positive annuity income level.

Introduce a new value function:

$$\begin{aligned}\overline{J}_t\left(\overline{W}_t, X_t\right) &= \max_{\left\{w_t,\overline{C}_t\right\}} \left\{u(\overline{C}_t) + \beta \cdot \mathbb{E}_t\left[p_{x+t,t} \cdot \overline{J}_{t+1}\left(\overline{W}_{t+1}, X_{t+1}\right)\right]\right\} \\ &\text{s.t.} \\ \overline{W}_{t+1} &= \left(\overline{W}_t + 1 - \overline{C}_t\right) \cdot \left(1 + r^{rf} + w_t \cdot \left(r_t - r^{rf}\right)\right) \\ \overline{C}_t &\leq \overline{W}_t + 1.\end{aligned}$$

For $t = MA - x$ this new value function satisfies $\overline{J}_{MA-x}\left(\overline{W}_{MA-x}, X_{MA-x}\right) = \frac{\left(\overline{W}_{MA-x}+1\right)^{1-\gamma}}{1-\gamma}$. At time $t = MA - x - 1$ the expression for $\overline{J}_t\left(\overline{W}_t, X_t\right)$ corresponds exactly to the maximization part in (6.26), hence we have:

$$J_t\left(W_t, A_t, X_t\right) = A_t^{1-\gamma} \cdot J_t\left(\overline{W}_t, 1, X_t\right) = A_t^{1-\gamma} \cdot \overline{J}_t\left(\overline{W}_t, X_t\right). \tag{6.27}$$

Using (6.27) instead of (6.25) the exact same logic can be repeated back until $t = s$. Hence, by solving the maximization problem:

$$\begin{aligned}\overline{J}_t\left(\overline{W}_t, X_t\right) &= \max_{\left\{w_t,\overline{C}_t\right\}} \left\{u(\overline{C}_t) + \beta \cdot \mathbb{E}_t\left[p_{x+t,t} \cdot \overline{J}_{t+1}\left(\overline{W}_{t+1}, X_{t+1}\right)\right]\right\} \\ &\text{s.t.} \\ \overline{W}_{t+1} &= \left(\overline{W}_t + 1 - \overline{C}_t\right) \cdot \left(1 + r^{rf} + w_t \cdot \left(r_t - r^{rf}\right)\right) \\ \overline{C}_t &\leq \overline{W}_t + 1,\end{aligned}$$

which has only a single state variable, it is possible to obtain the level of the value function, consumption, and other variables of interest by using the appropriate function, e.g., $J_t\left(W_t, A_t, X_t\right) = A_t^{1-\gamma} \cdot \overline{J}_t\left(\overline{W}_t, X_t\right)$ and $C_t\left(W_t, A_t, X_t\right) = A_t \cdot \overline{C}_t\left(W_t/A_t, X_t\right)$. Notice that the normalization with respect to A_t is equivalent with the normalization with respect to B_t, because by definition for time $t > s + d$ the value of A_t is equal to the value of B_t.

Next, consider the problem with a deferred annuity where the annuity is not yet paying, i.e., we consider the problem at time $s < t < s + d - 1$ with $d > 1$. We have that the utility is given by:

$$\begin{aligned}J_t\left(W_t, A_t = 0, B_t, X_t\right) &= \max_{\{w_t,C_t\}} \left\{u(C_t) + \beta\mathbb{E}_t\left[p_{x+t,t}J_{t+1}\left(W_{t+1}, A_{t+1}, B_{t+1}, X_{t+1}\right)\right]\right\} \\ &\text{s.t.} \\ W_{t+1} &= \left(W_t - C_t\right) \cdot \left(1 + r^{rf} + w_t \cdot \left(r_t - r^{rf}\right)\right) \\ C_t &\leq W_t.\end{aligned}$$

Using equation (6.27) for time $t = s + d$ and equation (6.29) for $t < s + d$, we can rewrite the utility function as:

$$\begin{aligned}
J_t\left(W_t, A_t{=}0, B_t, X_t\right) &= \max_{\{w_t, C_t\}} \left\{u(C_t) + \beta \mathbb{E}_t\left[p_{x+t,t} J_{t+1}\left(W_{t+1}, A_{t+1}, B_{t+1}, X_{t+1}\right)\right]\right\} \\
&\quad \text{s.t.} \quad (6.20)\text{-}(6.23) \\
&= \max_{\{w_t, \overline{C}_t\}} \Big\{ B_t^{1-\gamma} u(\overline{C}_t) \\
&\qquad + \beta B_{t+1}^{1-\gamma} \mathbb{E}_t\big[p_{x+t,t} J_{t+1}\big(\overline{W}_{t+1}, \overline{A}_{t+1}, 1, X_{t+1}\big)\big]\Big\} \\
&\quad \text{s.t.} \quad (6.20),\ (6.22'),\ (6.23'),\ \overline{W}_t{=}\frac{W_t}{B_t},\ \overline{C}_t{=}\frac{C_t}{B_t},\ \text{and } \overline{A}_t{=}\frac{A_t}{B_t} \\
&= B_t^{1-\gamma} \max_{\{w_t, \overline{C}_t\}} \Big\{u(\overline{C}_t) + \beta \mathbb{E}_t\big[p_{x+t,t} J_{t+1}\big(\overline{W}_{t+1}, \overline{A}_{t+1}, 1, X_{t+1}\big)\big]\Big\} \\
&\quad \text{s.t.} \quad (6.20'),\ (6.22'),\ \overline{W}_t{=}\frac{W_t}{B_t},\ \overline{C}_t{=}\frac{C_t}{B_t},\ \text{and } \overline{A}_t{=}\frac{A_t}{B_t} \qquad (6.28)
\end{aligned}$$

Define a new value function:

$$\begin{aligned}
\overline{J}_t\left(\overline{W}_t, \overline{A}_t = 0, X_t\right) &= \max_{\{w_t, \overline{C}_t\}} \left\{u(\overline{C}_t) + \beta \cdot \mathbb{E}_t\left[p_{x+t,t} \cdot \overline{J}_{t+1}\left(W_{t+1}, \overline{A}_{t+1}, X_{t+1}\right)\right]\right\} \\
&\quad \text{s.t.} \\
\overline{W}_{t+1} &= \left(\overline{W}_t - \overline{C}_t\right) \cdot \left(1 + r^{rf} + w_t \cdot \left(r_t - r^{rf}\right)\right) \\
\overline{C}_t &\leq \overline{W}_t.
\end{aligned}$$

This expression for $\overline{J}_t\left(\overline{W}_t, \overline{A}_t = 0, X_t\right)$ corresponds exactly with the maximization part in (6.28). Hence, we have $C_t(W_t, B_t, X_t) = B_t \cdot \overline{C}_t(\overline{W}_t, X_t)$ and:

$$\begin{aligned}
J_t\left(W_t, A_t = 0, B_t, X_t\right) &= B_t^{1-\gamma} \cdot J_t\left(\overline{W}_t, \overline{A}_t = 0, 1, X_t\right) \\
&= B_t^{1-\gamma} \cdot \overline{J}_t\left(\overline{W}_t, \overline{A}_t = 0, X_t\right). \qquad (6.29)
\end{aligned}$$

Now consider the moment of purchasing an immediate annuity. We have that the utility of the individual at time s is given by:

$$\begin{aligned}
J_s\left(W_s, A_s = 0, X_s\right) &= \max_{\{w_s, a_s, C_s\}} \left\{u(C_s) + \beta \cdot \mathbb{E}_s\left[p_{x+s,s} J_{s+1}\left(W_{s+1}, A_{s+1}, X_{s+1}\right)\right]\right\} \\
&\quad \text{s.t.} \\
W_{s+1} &= (W_s - C_s) \cdot (1 - a_s) \cdot \left(1 + r^{rf} + w_s \cdot \left(r_s - r^{rf}\right)\right) \\
A_{s+1} &= \frac{W_s - C_s}{V_s(A^{(1)}_{x+s,s})} \cdot a_s \\
C_s &\leq W_s.
\end{aligned}$$

Notice that the state variable of the annuity income at time s can be dropped, since it is by definition equal to zero, because the individual purchases only annuities at time s.

Define a variable with a hat by the corresponding variable without a hat divided by the wealth level at that moment.[23] We can rewrite the utility function as:

$$
\begin{aligned}
J_s\left(W_s, X_s\right) = & \max_{\{w_s, a_s, C_s\}} \left\{u(C_s) + \beta \cdot \mathbb{E}_s\left[p_{x+s,s} \cdot J_{s+1}\left(W_{s+1}, A_{s+1}, X_{s+1}\right)\right]\right\} \\
& \text{s.t. } (6.20)\text{-}(6.22) \\
& = \max_{\left\{w_s, a_s, \widehat{C}_s\right\}} \Big\{W_s^{1-\gamma} \cdot u(\widehat{C}_s) + \beta \cdot A_{s+1}^{1-\gamma} \cdot \mathbb{E}_s\left[p_{x+s,s} \cdot \right. \\
& \qquad \overline{J}_{s+1}\left(\frac{(1-a_s)V_s(A^{(1)}_{x+s,s})}{a_s}\left(1+r^{rf}+w_s\left(r_s-r^{rf}\right)\right), X_{s+1}\right)\Big]\Big\} \\
& \text{s.t. } \widehat{C}_s = C_s / W_s \\
& = W_s^{1-\gamma} \cdot \max_{\left\{w_s, a_s, \widehat{C}_s\right\}} \left\{u(\widehat{C}_s) + \beta \cdot \left(\frac{\left(1-\widehat{C}_s\right) \cdot a_s}{V_s(A^{(1)}_{x+s,s})}\right)^{1-\gamma} \cdot \mathbb{E}_s\left[p_{x+s,s} \cdot \right.\right. \\
& \qquad \overline{J}_{s+1}\left(\frac{(1-a_s)V_s(A^{(1)}_{x+s,s})}{a_s}\left(1+r^{rf}+w_s \cdot \left(r_s-r^{rf}\right)\right), X_{s+1}\right)\Big]\Big\} \\
& \text{s.t. } \widehat{C}_s = C_s / W_s
\end{aligned} \tag{6.30}
$$

Define a new value function:

$$
\begin{aligned}
\widehat{J}_s(A_s = 0, X_s) = & \max_{\left\{w_s, a_s, \widehat{C}_s\right\}} \left\{u(\widehat{C}_s) + \beta \cdot \left(\frac{\left(1-\widehat{C}_s\right) \cdot a_s}{V_s(A^{(1)}_{x+s,s})}\right)^{1-\gamma} \cdot \mathbb{E}_s\left[p_{x+s,s} \cdot \right.\right. \\
& \qquad \overline{J}_{s+1}\left(\frac{(1-a_s)V_s(A^{(1)}_{x+s,s})}{a_s}\left(1+r^{rf}+w_s\left(r_s-r^{rf}\right)\right), X_{s+1}\right)\Big]\Big\}
\end{aligned} \tag{6.31}
$$

$$
\begin{aligned}
& \text{s.t.} \\
& \widehat{C}_s \leq 1.
\end{aligned}
$$

This expression for $\widehat{J}_s(A_s = 0, X_s)$ corresponds exactly to the maximization part in

[23] Assuming $W_t > 0$ for $t \leq s$, i.e., the individual has, at times before the purchase of an annuity a positive wealth level. In case $W_t = 0$ for some $t \leq s$ the consumption level at that time is equal to zero, due to the budget constraint and hence $J_t(0) = -\infty$.

(6.30). Hence, we have that $C_s(W_s, A_s = 0, X_s) = \widehat{C}_s(A_s = 0, X_s) \cdot W_s$ and:

$$J_s(W_s, A_s = 0, X_s) = W_s^{1-\gamma} \cdot J_s(1, A_s = 0, X_s)) = W_s^{1-\gamma} \cdot \widehat{J}_s(A_s = 0, X_s). \tag{6.32}$$

This implies that for $t = s$ the optimal consumption level is a linear function of the wealth level, the optimal fraction of after consumption wealth invested in annuities is independent of the wealth level, and the after consumption and annuitized wealth invested in equities is independent of the wealth level.

Finally, we consider the time before the purchasing of an annuity, i.e. at time $t < s$. First, consider the problem at time $t = s - 1$:

$$\begin{aligned} J_t\left(W_t, A_t = 0, X_t\right) = &\max_{\{w_t, C_t\}} \left\{u(C_t) + \beta \mathbb{E}_t\left[p_{x+t,t} J_{t+1}\left(W_{t+1}, A_{t+1} = 0, X_{t+1}\right)\right]\right\} \\ &\text{s.t.} \\ W_{t+1} =& \left(W_t - C_t\right) \cdot \left(1 + r^{rf} + w_t \cdot \left(r_t - r^{rf}\right)\right) \\ A_{t+1} =& A_t \\ C_t \leq& W_t. \end{aligned} \tag{6.33}$$

Using equation (6.32), we can rewrite the utility function as:

$$\begin{aligned} J_t\left(W_t, A_t = 0, X_t\right) = &\max_{\{w_t, C_t\}} \left\{u(C_t) + \beta \mathbb{E}_t\left[p_{x+t,t} J_{t+1}\left(W_{t+1}, A_{t+1} = 0, X_{t+1}\right)\right]\right\} \\ &\text{s.t. (6.20)-(6.22)} \\ &= W_t^{1-\gamma} \cdot \max_{\left\{w_t, \widehat{C}_t\right\}} \Big\{u(\widehat{C}_t) + \beta \cdot \left(1 - \widehat{C}_t\right)^{1-\gamma} \cdot \mathbb{E}_t\big[p_{x+t,t} \\ &\quad \left(1 + r^{rf} + w_t\left(r_t - r^{rf}\right)\right)^{1-\gamma} J_{t+1}\left(1, A_{t+1} = 0, X_{t+1}\right)\big]\Big\} \\ &\text{s.t. } \widehat{C}_t = C_t / W_t \end{aligned} \tag{6.34}$$

Define a new value function:

$$\begin{aligned} \widehat{J}_t(A_t = 0, X_t) = &\max_{\{w_t, C_t\}} \Big\{u(\widehat{C}_t) + \beta \cdot \mathbb{E}_t\big[p_{x+t,t} \cdot \left(1 - \widehat{C}_t\right)^{1-\gamma} \cdot \\ &\left(1 + r^{rf} + w_t\left(r_t - r^{rf}\right)\right)^{1-\gamma} \widehat{J}_{t+1}(A_{t+1} = 0, X_{t+1})\big]\Big\} \end{aligned} \tag{6.35}$$

$$\begin{aligned} &\text{s.t.} \\ A_{t+1} =& A_t \\ \widehat{C}_t \leq& 1. \end{aligned}$$

This expression for $\widehat{J}_t(A_t = 0, X_t)$ corresponds exactly to the maximization part in (6.34). Hence, we have:

$$J_t\left(W_t, A_t = 0, X_t\right) = W_t^{1-\gamma} \cdot J_t\left(1, A_t = 0, X_t\right) = W_t^{1-\gamma} \cdot \widehat{J}_t(A_t = 0, X_t). \quad (6.36)$$

Using equation (6.36) instead of equation (6.32), the exact same logic can be repeated back an arbitrary number of periods. Hence, we have that $J_t(W_t, A_t = 0, X_t) = W_t^{1-\gamma} \cdot \widehat{J}_t(A_t = 0, X_t)$ and $C_t(W_t, A_t = 0, X_t) = \widehat{C}_t(A_t = 0, X_t) \cdot W_t$. This implies that for $t \leq s$ the optimal consumption level is a linear function of the wealth level, and the fraction of wealth invested in annuities (at time s) and the fraction of wealth invested in equities (at time $t \leq s$) is independent of the wealth level.

The individual's optimal choice for problem (6.33) can be determined by setting the first order conditions of the value function given in equation (6.35) equal to zero. However, in Appendix 6.B.3 we will use an alternative solution method, using $J_t(W_t, A_t = 0, X_t) = W_t^{1-\gamma} \cdot \widehat{J}_t(A_t = 0, X_t)$ and $C_t(W_t, A_t = 0, X_t) = \widehat{C}_t(A_t = 0, X_t) \cdot W_t$ which we obtained from the normalizations.

6.B.2 The optimal deferred annuity decision

The individual's lifetime investment and consumption problem is solved via backwards dynamic programming. In order to solve the problem we first determine the optimal individual's lifetime choice (i.e., the yearly fraction of wealth in equities and the yearly consumption) conditional on $a_0(d)$ (i.e., conditional on currently purchasing a deferred annuity) for a equally spaced grid of $a_0(d)$ between zero and one.

We first determine the optimal life-cycle choice, conditional on an annuity income stream. The individual is assumed to have no bequest motive, which implies that the individual's optimal consumption level at the last period is to consume all his, after annuity income, wealth. Let $\overline{W}_t = \frac{W_t}{B_{s+1}}$ be the normalized wealth level, and let $\overline{A}_t = \frac{A_t}{B_{s+1}}$ be the normalized annuity income which equals one if the deferred annuity has a payment at time t and zero otherwise, then assuming $d \leq MA - x$ the individual's time-$(MA - x)$ value function is given by:

$$J_{MA-x}\left(MA, \overline{W}_{MA-x}, 1, X_{MA-x}\right) = \frac{\left(\overline{W}_{MA-x} + 1\right)^{1-\gamma}}{1-\gamma}.$$

Similar to Koijen, Nijman, and Werker (2009), at any intermediate point in time the lifetime utility function satisfies the Bellman equation:

$$\begin{aligned}
&\overline{J}_t\left(x+t,\overline{W}_t,\overline{A}_t,X_t\right) = \\
&\max_{\{w_t,\overline{C}_t\}}\left\{\frac{\left(\overline{C}_t\right)^{1-\gamma}}{1-\gamma}+\beta\cdot\mathbb{E}_t\left[p_{x+t,t}\cdot\overline{J}_{t+1}\left(x+t+1,\overline{W}_{t+1},\overline{A}_{t+1},X_{t+1}\right)\right]\right\} \\
\text{s.t. }&(6.20)\text{-}(6.22),
\end{aligned} \tag{6.37}$$

At each point in time there are two control variables, $\left(\overline{C}_t,w_t\right)$. Therefore, similar to Koijen, Nijman, and Werker (2009), to determine the optimal values of the control variables, at each point in time $t\geq 0$ we need to set the first order derivatives of Bellman equation with respect to the control variables equal to zero:

$$0=\mathbb{E}_t\left[p_{x+t,t}\cdot\left(\overline{C}^{\star}_{t+1}\right)^{-\gamma}\cdot\left(r_t-r^{rf}\right)\right], \tag{6.38}$$

$$\overline{C}^{\star}_t=\left(\beta\cdot\mathbb{E}_t\left[p_{x+t,t}\cdot\left(\overline{C}^{\star}_{t+1}\right)^{-\gamma}\cdot\left(1+r^{rf}+w^{\star}_t\cdot\left(r_t-r^{rf}\right)\right)\right]\right)^{\frac{1}{-\gamma}}. \tag{6.39}$$

Equation (6.38) is the first order condition of (6.37) with respect to the fraction of liquid wealth invested in equity and equation (6.39) is the first order condition of (6.37) with respect to the consumption level.

To solve the individual's lifetime investment and consumption problem we use a grid of the endogenous state variable after-consumption normalized wealth, $\widetilde{W}_t=\overline{W}_t+\overline{A}_t-\overline{C}_t$. Following Carroll (2006) we use an after-consumption wealth space with a triple exponential growth rate between the grid points between 0.001 and 100 and the grid point 0. To obtain results within reasonable time, we compute the expectations through a regression, similar to the simulation method proposed by Longstaff and Schwartz (2001) for pricing American-style options. First, we solve $w^{\star}_t$ using equation (6.38), next we solve $\overline{C}^{\star}_t$ using equation (6.39). To solve $w^{\star}_t$ based on (6.38), we follow Koijen, Nijman, and Werker (2009). We define H "test portfolios" with different fractions of wealth invested in equities, i.e., each test portfolio is characterized by its fraction of wealth invested in equities. We use an equally spaced grid of the fraction of liquid wealth invested in equities, i.e., $w_t(h)\in\{0,1/(H-1),\ldots,1\}$. Then, for a given test portfolio, we parameterize the conditional expectation as a function of the exogenous state variables in order to solve equation (6.38).[24] Let $\overline{C}^{\star}_{t+1}(h)$ be the optimal normalized consumption

[24] Following Koijen, Nijman, and Werker (2009) we compute the expectation through regressions

level in the following year, conditional on the corresponding fraction of liquid assets invested in equities ($w_t(h)$). The exogenous state variables at time t depend on the information known to the investor at time t. We simulate the equity return and the survival probabilities in the CBD model and we parameterize the following simulated expectation:

$$\mathbb{E}_t\left[p_{x+t,t}\cdot\left(\overline{C}^{\star}_{t+1}(h)\right)^{-\gamma}\cdot\left(r_t-r^{rf}\right)\right]=X_t^P\cdot\theta_t\left(\widetilde{W}_t,w_t(h)\right),\ h=1,\ldots,H,\ (6.40)$$

where $\theta_t\left(\widetilde{W}_t,w_t(h),\right)$ are the parameters to be estimated using least squares, and X_t^P is a polynomial expansion in the exogenous state variables.[25] We use $N=5,000$ simulated trajectories and $M=130$ points. Since we know the optimal consumption policy at time $t+1$ only at the endogenous grid points, we interpolate the consumption policy linearly for intermediate values. Let K be the dimension of X_t^P, and let $\theta_{t,k}\left(\widetilde{W}_t,w_t(h)\right)$ be the regression coefficients of $\theta_t\left(\widetilde{W}_t,w_t(h)\right)$ corresponding to the k^{th} component of X_t^P. To solve the equality in (6.38) we parameterize the regression parameters on a polynomial basis of the second degree in the asset weights:

$$\theta_{t,k}\left(\widetilde{W}_t,w_t(h)\right)=X_t^h\cdot\psi_{k,t}\left(\widetilde{W}_t\right),\text{for } k=1,\ldots,K, \quad (6.41)$$

where $\psi_{t,k}\left(\widetilde{W}_t\right)$ are the parameters to be estimated using least squares, and $X_t^h=\left[1\ w_t(h)\ (w_t(h))^2\right]$ is a polynomial basis in the fraction of liquid wealth annuitized in the test portfolios. Notice that we use a polynomial basis of the second degree instead of the first degree as is done in Koijen, Nijman, and Werker (2009), since this provides a better fit of the regression parameters. Given the parametrization in equation (6.41) we determine the optimal fraction of wealth invested in equities from setting the right hand side of equation (6.40) equal to zero, using projection coefficient of the polynomial expansion (i.e., using $\psi_{t,k}\left(\widetilde{W}_t\right)$).[26]

for a grid of the endogenous state variables. Even though it is theoretical possible to use a grid of exogenous and endogenous state variables, in order to properly parameterize the conditional expectation it requires a very large regression, which makes it computationally unattractive.

[25]We use a polynomial expansion of the second degree in $\exp\left(\widehat{\mu}_t^{(1)}\right)$, $\exp\left(\widehat{\mu}_t^{(2)}\right)$, $\exp\left(\widehat{C}_t^{(1,1)}\right)$, $\exp\left(\widehat{C}_t^{(1,2)}\right)$, $\exp\left(\widehat{C}_t^{(2,2)}\right)$, $\exp\left(k_t^{(1)}\right)$, and $\exp\left(k_t^{(2)}\right)$, including cross-terms in the polynomial expansion, to capture all relevant information on the exogenous state variables. We tried several polynomial expansions of the state variables. This specification turned out to be the most accurate, i.e., had the lowest sum of squared error. We take the exponent of the state variables, because the future survival probabilities are exponential transformations of the state variables.

[26]Notice that using this method does not incorporate the short-selling constraints. The short-

Next, we determine the optimal consumption level by solving equation (6.39), given the obtained optimal fraction of wealth invested in equities, using the above described method. In order to avoid to make N simulations for each trajectory, we parameterize the conditional expectation in equation (6.39) as a polynomial expansion of the exogenous state variables X_t^P, which depends on the information known to the investor at time t:

$$\mathbb{E}_t\left[p_{x+t,t}\cdot\left(\overline{C}_{t+1}^{\star}\right)^{-\gamma}\cdot\left(1+r^{rf}+w_t^{\star}\cdot\left(r_t-r^{rf}\right)\right)\right]=\exp\left(X_t^P\cdot\nu_t\left(\widetilde{W}_t\right)\right), \tag{6.42}$$

where $\nu_t\left(\widetilde{W}_t\right)$ are the parameters to be estimated using least squares. We use linear regression after taking the logarithm of the expectation, in order to ensure that the conditional expectation is strictly positive implying that the consumption is strictly positive. The wealth level at time t follows from $\overline{W}_t=\widetilde{W}_t-\overline{A}_t+\overline{C}_t$.

Finally, we have to determine the optimal fraction of wealth invested in a deferred annuity. To determine the optimal fraction of wealth currently (at time $s=0$) invested in deferred annuities we calculate the expected utility for different portfolios, conditional on an individual's initial wealth level. The portfolios differ in the fraction of annuitized wealth, and hence their annuity income level. Conditional on the purchase of a deferred annuity with a deferral period of d years, $a_0^{\star}(d)$ the optimal fraction of annuitized wealth is then determined by:

$$a_0^{\star}(d)=\underset{a_0(d)}{\operatorname{argmax}}\ J_0\left(W_0(a_0(d)),A_0(a_0(d)),B_0(a_0(d)),X_0\right), \tag{6.43}$$

where $W_0(a_0(d))$, $A_0(a_0(d))$, $B_0(a_0(d))$ are the values of the endogenous state variables conditional on $a_0(d)$. The optimal fraction of annuitized wealth is thus obtained by comparing the expected lifetime utility of an individual, given the optimal consumption and investment choices, conditional on the fraction of annuitized wealth.

6.B.3 The optimal postponed annuity level

In this appendix we describe the method to obtain the optimal annuity decision when the individual postpones the purchase of an immediate annuity. Contrary to

selling constraints implies that the fractions of wealth invested in equities should be between zero and one. Therefore, we investigate whether the corner solutions (i.e., $w_t^{\star}=0$ and $w_t^{\star}=1$) are optimal. In case both roots are in the $[0,1]$ interval, we select the one with the highest utility level.

currently purchasing a deferred annuity, when postponing the purchase of immediate annuities, the decision may depend on the state variables. Let s be the number of years until the annuity is purchased, hence the individual purchases an immediate annuity ($d = 1$) at time s. In this case B_t does not play a role, and hence the endogenous state variables at time t are fully captured by A_t and W_t.

To solve the optimal consumption and investment choices using the backwards induction algorithm, we distinguish three types of decision moments, namely:

i) *After time s.* The optimal decisions are obtained for a grid of endogenous state variables. Using the method described in Appendix 6.B.2 we obtain optimal investment and consumption choices conditional on the wealth level, the annuity income level, and the exogenous state variables. We normalize by the annuity income level, the normalizations are described in Appendix 6.B.1. We recover the original state variables and $C_t^\star$ by multiplying the coreresponding normalized variable with the annuity income level.

ii) *At time s.* At this time the individual has to determine the optimal annuity income level, besides the optimal consumption and investment choices. The method to obtain these optimal choices is described in the remainder of this section.

iii) *Before time s.* The optimal decisions are obtained for a given wealth level. Using the method described in Appendix 6.B.2 the optimal investment and consumption choices are obtained, conditional on the exogenous state variables and the wealth level.

At time $t \leq s$ we solve the original problem to determine the optimal choice as a function of the wealth level state variables. Because we know the that there is a linear relation between the wealth level and consumption, and that the optimal fraction of wealth invested in equity or annuities does not depend on the wealth level, we know the optimal choices of the individual for any wealth level when we calculate it only for one positive wealth level. The advantage of this second method is that the formula of the first order conditions are the same as the formula of the first order conditions for the optimal deferred annuity decision, as given in equation (6.38) and (6.39). Hence, we can also use the method described in Appendix 6.B.2 to determine the individual's optimal choice when the individual has not yet purchased an annuity. Therefore, in Appendix 6.B.3 we determine the optimal $C_t(W_t, A_t = 0, X_t)$ and the optimal fraction of after-consumption wealth invested in equity given that $W_t - C_t = 1$. Let $\underline{W}_t$ be the wealth level at time t which leads to an after consumption wealth of 1 at time t, i.e., $\underline{W}_t - C_t^\star(\underline{W}_t, A_t = 0, X_t) = 1$. To obtain the optimal consumption level for a $W_t > 0$ we use:

$$\begin{aligned} C_t^\star(W_t, A_t = 0, X_t) =& C_t^\star\left(\frac{W_t}{1 + C_t^\star(\underline{W}_t, A_t = 0, X_t)}\right.\\ &\left.\cdot \left(1 + C_t^\star(\underline{W}_t, A_t = 0, X_t)\right), A_t = 0, X_t\right)\\ =& \frac{W_t}{1 + C_t^\star(\underline{W}_t, A_t = 0, X_t)} \cdot C_t^\star(\underline{W}_t, A_t = 0, X_t)\\ =& \frac{C_t^\star(\underline{W}_t, A_t = 0, X_t)}{1 + C_t^\star(\underline{W}_t, A_t = 0, X_t)} \cdot W_t. \end{aligned} \tag{6.44}$$

The approach to obtain the optimal decisions at time s is an extension of the method explained in Appendix 6.B.2, by adjusting the first order conditions for the annuity decision and including the first order condition for the optimal fraction of annuitized wealth. To obtain the optimal decisions at time s we have three control variables, namely $(C_s, w_s, a_s(1))$. An increase in a_s and thus A_t leads to an increase in an individual's income level at time $t > s + d$ of the same size and, thus, impacts his wealth level at time t. Moreover, an increase in A_t leads to an increase in A_{t+1} of the same magnitude, i.e., the annuity income increases not only for the following year, but for all future years.

In order to avoid over-notation denote $J_t = J_t(x + t, W_t, A_t, X_t^P)$. Using $\frac{\partial J_t}{\partial W_t} = (C_t^\star)^{-\gamma}$ we have that:

$$\frac{\partial J_t}{\partial A_t} = (C_t^\star)^{-\gamma} + \beta \cdot \mathbb{E}_t\left[p_{x+t,t} \cdot \frac{\partial J_{t+1}}{\partial A_{t+1}}\right] \text{ for } t \geq s + 1.$$

Moreover,

$$\begin{aligned} &\frac{\partial J_{s+1}}{\partial A_{s+1}} / \frac{\partial J_{s+1}}{\partial W_{s+1}} = \\ &\mathbb{E}_{s+1}\left[1 + \left(p_{x+s+1,s+1}\beta\left(\frac{C_{s+2}^\star}{C_{s+1}^\star}\right)^{-\gamma} \cdot \left(1 + p_{x+s+2,s+2}\beta\left(\frac{C_{s+3}^\star}{C_{s+2}^\star}\right)^{-\gamma} \cdot (\ldots)\right)\right)\right] \end{aligned} \tag{6.45}$$

Notice that both $\mathbb{E}_{s+1}\left[1+\left(p_{x+s+1,s+1}\beta\left(\frac{C_{s+2}^\star}{C_{s+1}^\star}\right)^{-\gamma}\left(1 + p_{x+s+2,s+2}\beta\left(\frac{C_{s+3}^\star}{C_{s+2}^\star}\right)^{-\gamma}(\ldots)\right)\right)\right]$ and, in the normalized problem $\overline{J}_{s+1}\left(\frac{1-a_s(1)}{a_s(1)}\left(1 + r^{rf} + w_s\left(r_s - r^{rf}\right)\right) V_s(A_{x+s,s}^{(1)})\right)$ depend on time $s+1$ expectation of a function of the optimal consumption levels after time $s+1$. Hence, calculating (6.45) instead of the normalized problem would require the same calculations.

For the original problem, we set the first order derivatives at time s of the Jacobian with respect to each of the three control variables equal to zero:

$$
\begin{aligned}
0 =&\mathbb{E}_s\left[p_{x+s,s}\cdot\left(C^\star_{s+1}\right)^{-\gamma}\cdot(1-a^\star_s(1))\cdot\left(r_s-r^{rf}\right)\right],\\
0 =&\mathbb{E}_s[p_{x+s,s}\cdot\left(C^\star_{s+1}\right)^{-\gamma}\cdot(\frac{1}{V_s(A^{(1)}_{x+s,s})}\cdot\frac{\partial J_{s+1}}{\partial A_{s+1}}/\frac{\partial J_{s+1}}{\partial W_{s+1}}\\
&-(1+r^{rf}+w^\star_s\cdot\left(r_s-r^{rf}\right)))],\\
(C^\star_s)^{-\gamma} =&\beta\cdot\mathbb{E}_s\left[p_{x+s,s}\cdot\left(C^\star_{s+1}\right)^{-\gamma}\cdot\left(a^\star_s(1)\cdot\frac{1}{V_s(A^{(1)}_{x+s,s})}\cdot\frac{\partial J_{s+1}}{\partial A_{s+1}}/\frac{\partial J_{s+1}}{\partial W_{s+1}}\right.\right.\\
&\left.\left.+(1-a^\star_s(1))\cdot\left(1+r^{rf}+w^\star_s\cdot\left(r_s-r^{rf}\right)\right)\right)\right],
\end{aligned}
$$

where $C^\star_s$, $w^\star_s$, and $a^\star_s(1)$ are the time-s optimal consumption, fraction of liquid wealth invested in equity, and fraction of wealth invested in annuities, respectively.

To solve the choices at time s we define H^2 test portfolios. These test portfolios are characterized by the fraction of after-consumption wealth invested in annuities ($a_s(1,h_1)\in\{0,1/(H-1),\ldots,1\}$, for $h_1\in H$), and the fraction of after-annuitized liquid wealth invested in equity ($w_s(h_2)\in\{0,1/(H-1),\ldots,1\}$, for $h_2\in H$). Hence, each test portfolio is characterized by $(a_s(1,h_1),w_s(h_2))$. Let $h=(h_1,h_2)$, and $C^\star_{s+1}(h)$ be the optimal consumption level at time $s+1$ corresponding to test portfolio h, and $J_{s+1}(h)$ the value of the Bellman at time $s+1$ corresponding to test portfolio h. We generalize equation (6.40), to solve the investment decision at time s we first parameterize for every $h\in H^2$:

$$
\begin{aligned}
X^P_s\cdot\theta^w_{s,k}\left(a_s(1,h_1),w_s(h_2)\right) =&\mathbb{E}_s\left[p_{x+s,s}\left(C^\star_{s+1}(h)\right)^{-\gamma}(1-a_s(1,h_1))\left(r_s-r^{rf}\right)\right]\\
X^P_s\cdot\theta^a_{s,k}\left(a_s(1,h_1),w_s(h_2)\right) =&\mathbb{E}_s[p_{x+s,s}\left(C^\star_{s+1}(h)\right)^{-\gamma}\left(-1-r^{rf}-w^\star_s\left(r_s-r^{rf}\right)\right.\\
&\left.+\frac{\partial J_{s+1}(h)}{V_s(A^{(1)}_{x+s,s})\cdot\partial A_{s+1}}/\frac{\partial J_{s+1}(h)}{\partial W_{s+1}}\right)].
\end{aligned}
$$

Furthermore, we generalize equation (6.41) by parameterizing the regression coefficients on a polynomial basis:

$$
\begin{aligned}
\theta^w_{s,k}\left(a_s(1,h_1),w_s(h_2)\right) &= X^h_s\cdot\psi^w_{s,k},\\
\theta^a_{s,k}\left(a_s(1,h_1),w_s(h_2)\right) &= X^h_s\cdot\psi^a_{s,k},
\end{aligned}
$$

where X_s^h is a polynomial expansion in the portfolio weights of the second degree.[27]

Finally, generalizing equation (6.42), we use the following parametrization to obtain the optimal consumption level:

$$\begin{aligned}\exp\left(X_s^P \cdot \nu_s\right) =& \mathbb{E}_s[p_{x+s,s} \cdot \left(C_{s+1}^\star\right)^{-\gamma} \cdot (a_s^\star(1) \cdot \frac{\partial J_{s+1}}{V_s(A_{x+s,s}^{(1)}) \cdot \partial A_{s+1}} / \frac{\partial J_{s+1}}{\partial W_{s+1}} \\ &+ (1 - a_s^\star(1)) \cdot \left(1 + r^{rf} + w_s^\star \cdot \left(r_s - r^{rf}\right)\right))].\end{aligned}$$

Given the above described technique we find the optimal annuity, consumption, and investment decision at time s, conditional on the exogenous state variables and a after-consumption wealth of one. The optimal investment decisions ($w_s^\star$, and $a_s^\star(1)$) are independent of the wealth level, and in order to recover the original optimal consumption level we use equation (6.44). Using the technique described in Appendix 6.B.2 we obtain the individual's optimal consumption and investment choices for any time when he does not purchase an annuity.

[27]When the optimal $w_s^\star$ or $a_s^\star(1)$ is outside the [0,1]-interval we determine the optimal corner solutions, i.e., $a_s^\star(1) = 1$ and we determine $w_s^\star$ conditional on $a_s(1) = 0$, $a_s^\star(1)$ conditional on $w_s^\star = 1$, and $a_s^\star(1)$ conditional on $w_s^\star = 0$, and select the one with the highest utility level. Notice that when $a_s^\star(1) = 1$, there is no liquid wealth after consumption, and hence the first order condition with respect to w_t equals zero, for all w_t.

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Langlevenrisico in levensverzekeringsproducten

Samenvatting

In dit proefschrift behandelen we enkele onderwerpen op het gebied van langlevenrisico. Het uitstaand risico van producten met langlevenrisico, met name door pensioenfondsen en levensverzekeraars, is groot. Een indicatie van de grote van de markt voor langlevenrisicoproducten is dat het vermogen van Nederlandse pensioenfondsen, volgens de OECD, 129.82% (ongeveer 750 miljard euro) van het bruto binnenlands product was in 2009.[28] Wereldwijd is het private pensioenvermogen opgelopen tot een waarde van 25 triljoen dollar aan het eind van 2009.[29] Al dit vermogen is opgebouwd om, na pensionering, voldoende vermogen te hebben om de rest van het leven te consumeren, het liefst op hetzelfde niveau als voor pensionering. Als we allen (gemiddeld) langer leven dan we momenteel verwachten hebben we ook meer vermogen nodig voor de consumptie tijdens ons pensioen.

Voor langlevenrisico wordt er vaak onderscheid gemaakt tussen een systematisch en een niet-systematisch deel. Het niet-systematische langlevenrisico is het risico dat, gegeven de jaarlijkse sterftekansen, in een portefeuille het aantal overlevenden ieder jaar onzeker is. Met de wet van de grote aantallen kan bewezen worden dat dit risico relatief verwaarloosbaar klein wordt als het aantal deelnemers in de portfolio maar groot genoeg is. Het systematisch langlevenrisico van een pensioenproduct is daarentegen onafhankelijk van het aantal deelnemers. Dit systematisch langlevenrisico komt doordat toekomstige sterftekansen momenteel onzeker zijn. In dit proefschrift focussen we op het systematisch langlevenrisico, omdat het niet-systematisch langlevenrisico klein is voor portefeuilles van pensioenfondsen met een redelijk aantal

[28]Bron: OECD Global Pension Statistics, http://stats.oecd.org/.

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deelnemers. Dit proefschrift behandelt verschillende manieren om het systematisch langlevenrisico voor pensioenfondsen en levensverzekeraars te bepalen en het optimaal beheren hiervan. Dit kan door middel van de manier van pensioenopbouw (met partnerpensioen op "risicobasis" of op "opbouwbasis"), natuurlijke hedge mogelijkheden en kapitaalmarktoplossingen.

In Hoofdstuk 2 geven we een overzicht van de bestaande literatuur op het gebied van langlevenrisico. We focussen hier op de modellen om de kansverdeling van toekomstige sterftekansen te voorspellen, verschillende gebruikte methodes om langlevenrisico te kwantificeren voor levensverzekeringsproducten en methodes om het langlevenrisico te beheersen. De modellen om de kansverdeling van toekomstige sterftekansen te voorspellen die worden beschreven zijn het Lee-Carter model, het Cairns-Blake-Dowd model en het P-splines model en uitbreidingen op de modellen. Deze modellen worden in de rest van het proefschrift ook gebruikt om de kansverdeling van toekomstige sterftekansen te modelleren.

In de literatuur bestaan er drie verschillende methoden om het langlevenrisico in levensverzekeringsproducten te kwantificeren. De methoden zijn met behulp van de onzekerheid in de verdisconteerde contante waarde van de verplichtingen, de onzekerheid in de toekomstige dekkingsgraad en de ruïne kans. Deze methodes hebben ieder hun voor- en nadelen en we gebruiken in de overige hoofdstukken van het proefschrift alle drie methoden om het langlevenrisico in levensverzekeringsproducten te kwantificeren. Bovendien laten we in dit hoofdstuk zien dat voor een Nederlands pensioenfonds met alleen 65-jarige mannen met ouderdomspensioenrechten het aantal deelnemers slecht 628 hoeft te zijn wil het systematisch langlevenrisico een groter effect hebben dan het niet-systematisch langlevenrisico. Voor een fonds met alleen 65-jarige vrouwen met ouderdomspensioenrechten is het systematisch langlevenrisico al groter dan het niet-systematische bij een aantal deelnemers van 317.

Tot slot behandelen we in Hoofdstuk 2 nog mogelijke manieren om het langlevenrisico in levensverzekeringsproducten te reduceren. De overige hoofdstukken van dit proefschrift dragen ook bij aan het onderzoek naar de efficiëntie van de risicoreductie van enkele van deze manieren. Een mogelijke manier is via kapitaalmarktoplossingen het risico over te hevelen aan andere partijen, met waarschijnlijk daarbij een betaling van een risicopremie. De producten in de markt van langlevenrisico kunnen vergelijkbaar zijn met de producten in de markt van rampenrisico welke verhandelt worden door middel van "catastrophe bonds," kortweg CAT bonds. Tussen januari 2008, toen de eerste langlevenrisicotransactie plaatsvond, en februari

2010 hebben er acht langlevenrisicotransacties plaatsgevonden. De nog jonge markt voor langlevenrisico, in tegenstelling tot de rampenrisicomarkt, is nog erg illiquide. Daardoor is het nog niet mogelijk om een marktprijs voor langlevenrisico te bepalen.

In Hoofdstuk 3 bepalen we de prijs van langlevenrisico voor verschillende levensverzekeringsproducten en de hoeveelheid benodigde bufferkapitaal voor deze producten. We doen dit door middel van een marktconsistente manier, namelijk de Cost of Capital methode. Deze methode wordt ook voorgesteld in Solvency II, het voorstel voor de nieuwe regelgeving ten aanzien van onder andere het benodigd kapitaal en waardering van verplichtingen voor verzekeraars in Europa. In de Cost of Capital methode moet de verzekeraar voldoende vermogen aanhouden om met ten miste 99.5% zekerheid over één jaar geen dekkingsgraad onder de 100% te hebben. De regelgever stelt dus verplicht dat de verzekeraar een buffer aanhoud ter groote van het verschil tussen het vereist vermogen en de (markt-) waarde van de verplichtingen. Vanwege het aanhouden van deze buffer, die niet vrij kan worden geïnvesteerd, wil de verzekeraar een vergoeding hiervoor. Deze vergoeding, een zogenoemde "Cost of Capital rate" vermenigvuldigd met de grootte van de buffer, is de risicopremie voor het langlevenrisico in de Cost of Capital methode. In deze methode is waarde van een levensverzekeringsproduct dus gelijk aan de verwachte contante waarde van de verplichtingen plus de risicopremie.

Levensverzekeringsproducten hebben typisch een lange looptijd die de berekening van de risicopremie compliceert. Voor de bepaling van de risicopremie is namelijk niet alleen de huidige grootte van de buffer nodig, welke zorgt dat de kans op onderdekking over één jaar kleiner is dan 0.5%, maar ook de verwachte waarde van alle toekomstige buffers. Bovendien hangt de dekkingsgraad in een volgend jaar af van de, op dat tijdstip geldende, waarde van de verplichtingen (op marktwaarde) en dus ook van de, op dat tijdstip, verwachte toekomstige buffers. Dit zorgt ervoor dat accurate bepaling van de waarde van verplichtingen met behulp van simulaties te veel computertijd in beslag zou nemen. Daarom ontwikkelen we een gesloten vorm benadering van de kansverdeling van de waarde van toekomstige betalingen van levensverzekeringsproducten. Met behulp van deze benaderde kansverdeling is het doen van simulaties voor de bepaling van de waarde van verplichtingen niet meer nodig en kan de waarde van de levensverzekeringsproducten snel worden bepaald.

De regelgever onderkent de problemen van het vaststellen van de benodigde buffer en de waarde van de verplichtingen met behulp van de Cost of Capital methode. Daarom stelt zij een vereenvoudiging voor om de berekeningen uit te voeren. De benodigde buffer voor het éénjaarsrisico kan in deze vereenvoudiging bepaald worden door middel van twee scenario's. Het ene scenario is de verwachte toekomstige sterftekansen en het andere scenario is een onmiddellijke en permanente daling in de verwachte toekomstige sterftekansen met 20%. De benodigde buffer voor het éénjaarsrisico is gelijk aan het verschil in de contante waaarde tussen deze twee scenario's. Verzekeraars hebben de keuze tussen een intern model te gebruiken volgens de Cost of Capital methode of deze benadering toe te passen bij de bepaling van de buffer en de waarde van de verplichtingen. In Hoofdstuk 3 vergelijken we de waarde van de verplichtingen en de benodigde buffer voor levensverzekeringsproducten voor een 65-jarige. Wij vinden dat de vereenvoudiging van de regelgever leidt tot een veel hogere waarde ven de verplichtingen en benodigde buffer dan wanneer deze berekend worden met behulp van ons voorgestelde interne model. Dit komt doordat het interne model uitgaat van het Lee-Carter model om toekomstige sterftekansen te voorspellen, waarbij het eenjarig risico veel kleiner is dan een reductie in de toekomstige sterftekansen binnen één jaar met 20% zullen afnemen. Daarnaast stelt de regelgever een aantal vereenvoudigingen met betrekking tot het interne model voor. We constateren dat de vereenvoudigingen een groot effect hebben op de waarde van de verplichtingen bij het verge-lijken van de waarde met behulp van het interne model met en zonder de voorgestelde vereenvoudigingen met betrekking tot het interne model. De voorgestelde vereenvoudigingen met betrekking tot het interne model leiden er in het algemeen toe dat de waarde van de verplichtingen worden onderschat in vergelijking tot de waarde van de verplichtingen zonder de vereenvoudigingen.

Hoofdstuk 4 onderzoekt de mogelijkheid om langlevenrisico voor pensioenfondsen te verlagen met behulp van de manier van opbouw van het pensioen. We vergelijken in dit hoofdstuk twee verschillende pensioencontracten. De deelnemers van het pensioenfonds bouwen in beide pensioencontracten ouderdomspensioen, welke betalingen heeft vanaf leeftijd 65 indien de deelnemer levend is, op. Het verschil tussen de pensioencontracten is dat het partnerpensioen, welke betalingen heeft wanneer de deelnemer is overleden maar de partner van de deelnemer nog levend is, in het ene contract op opbouwbasis is en in het andere contact op risicobasis. In Nederland, maar ook bijvoorbeeld in Amerika, kan je pensioen volgens de beide contracten opbouwen. Een eis van de regelgever is dat op het moment van pensionering de deelnemer het recht heeft, onder voorwaarde van de toestemming van de partner, om de ene vorm van pensioenrechten (deels) uit te ruilen voor de andere vorm van pensioenrechten. Deze uitruil moet minimaal actuarieel neutraal zijn, wat betekent dat de waarde van verplichten niet mag verminderen op het moment dat deelnemers pensioenrechten uitruilen.

De verplichting tot de actuarieel neutrale uitruilvoet op moment van uitruil leidt ertoe dat de toekomstige uitruilvoet onzeker is en afhangt van de evolutie van de toekomstige sterftekansen. Voor deelnemers die nog pensioenrechten opbouwen en pensioenrechten willen uitruilen leidt dit ertoe dat de rechten na uitruil momenteel onzeker zijn en afhangen van de ontwikkeling in de sterftekansen. Met andere woorden, indien een deelnemer in de toekomst pensioen wil uitruilen, zijn de rechten van de deelnemer afhankelijk van de evolutie inde sterftekansen en dus loopt de deelnemer langlevenrisico in zijn opgebouwde pensioenrechten. Dit geldt voor beide manieren van de pensioenopbouw. Voor het pensioenfonds is het langlevenrisico echter kleiner voor het pensioencontract waarbij de partnerpensioen wordt opgebouwd dan waarbij het partnerpensioen op risicobasis is. Dit betekent dat een pensioenfonds dat partnerpensioen heeft op risicobasis langlevenrisico kan verminderen door het pensioencontract aan te passen en het partnerpensioen op te bouwen. Deze verandering in het pensioencontract kan actuarieel neutraal door zodanig minder ouderdomspensioen op te bouwen dat, indien het partnerpensioen uitgeruild wordt, de deelnemer in verwachting evenveel ouderdomspensioen heeft als wanneer er alleen ouderdomspensioen wordt opgebouwd en partnerpensioen op risicobasis is.

In Hoofdstuk 5 bestuderen we het effect van verschillende manieren om langlevenrisico te reduceren op een portefeuille van levensverzekeringsproducten. We doen dit op basis van de ruïne kans, waarbij we naast langlevenrisico ook investeringsrisico meenemen. Het investeringsrisico, welke bestaat uit renterisico en aandelenrisico, heeft een substantieel effect op de ruïne kans, zelfs als de verzekeraar alleen in obligaties belegt en het renterisico afdenkt in de verwachte betalingen. Dit komt door het langlevenrisico, welke ervoor zorgt dat de hoogte van toekomstige betalingen momenteel onzeker is en dus zal er een mismatch zijn in de afdekking van het renterisico in de verwachte en de werkelijke betalingen.

Daarnaast kwantificeren we de effecten van de portefeuillesamenstelling (verhouding man-vrouw), productmix (verhouding ouderdomspensioen-partnerpensioen), het effect van overlijdingrisicoverzekeringen, welke een eenmalige uitbetaling hebben op het moment dat de verzekerde overlijdt en van “survival swaps”. De risicoreductie van de verschillende manieren is afhankelijk van het investeringsrisico. Een voorbeeld is dat het langlevenrisico in ouderdomspensioen volledig kan worden afgedekt met overlijdingrisicoverzekeringen als er geen onzekerheid is in de toekomstige rentetermijn structuur en deze vlak is. Echter, er blijft een substantieel langlevenrisico over wanneer men wel renterisico meeneemt in de berekening van de risicoreductie door middel van overlijdingrisicoverzekeringen opnemen in een portefeuille van ouderdomspensioenverplichtingen. Bovendien laten we zien dat er substantieel langlevenrisico overblijft indien er basisrisico is bij survival swaps. Het basisrisico ontstaat doordat de overlevingskansen van de survival swap zijn gebaseerd op de gehele populatie van een land, maar de verzekeraar pensioenverplichtingen heeft aan verzekerden, die andere (hogere) overlevingskansen hebben.

In Hoofdstuk 6 onderzoeken we het effect dat langlevenrisico heeft op de keuze voor individuen. In het bijzonder onderzoeken we het effect op de investeringkeuze, met name de keuze van annuïteiten, in een levensloopmodel. In de bestaande literatuur is een welbekend feit dat annuïteiten kunnen zorgen voor nutsverhoging, vanwege het element van risicodeling (als je langer leeft dan verwacht blijf je nog steeds de periodieke uitkeringen houden). Naast een onmiddellijke annuïteit, die zoals de naam al zegt meteen uitbetalingen heeft, bestaan er ook uitgestelde annuïteiten, die zoals de naam al doet vermoeden periodieke uitkeringen heeft na een bepaalde, vooraf vastgestelde, periode. We bekijken of het optimaal is voor een individu om op moment van pensionering (leeftijd 65) een annuïteit te kopen en welke annuïteit, of dat het optimaal is om een bepaalde periode te wachten alvorens een annuïteit te kopen. Een individu kan naast de annuïteit ook investeren in een risicovrij asset en in aandelen. Het voordeel van investeren in aandelen is dat het een risicopremie oplevert en dus een hoger verwacht rendement heeft dan het risicovrije asset. Als een individu de koop van de annuïteit uitstelt heeft hij langer liquide vermogen (annuïteiten kun je als individu namelijk alleen kopen, niet verkopen), een nadeel is echter dat de prijs in de toekomst momenteel onzeker is. De toekomstige prijs van annuïteiten is namelijk afhankelijk van de verwachte ontwikkeling van de overlevingskansen op het moment van aankoop.

We vinden dat langlevenrisico de keuzes van een individu op twee manieren beïnvloed. Ten eerste, vanwege langlevenrisico en daardoor de onzekerheid in toekomstige annuïtetsprijzen, is het nutverhogend voor individuen om op het moment van pensionering annuïteiten te kopen en dit niet uit te stellen tot oudere leeftijden. Hierin verschilt ons resultaat met de bestaande literatuur, welke geen langlevenrisico meeneemt. Ten tweede maakt langlevenrisico een uitgestelde annuïteit minder aantrekkelijk. De optimale annuïteit voor een 65 jarige is een uitgestelde annuïteit, waarbij de betalingen starten slechts enkele jaren na aanschaf. Desondanks, indien een individu een onmiddellijke annuïteit koopt in plaats van een uitgestelde, is het nutsverlies klein.

Dit proefschrift geeft verschillende bevindingen van het effect van langlevenrisico voor pensioenfondsen, verzekeraars en individuen. Met het toenemend bewustzijn in de pensioen- en verzekeringsector van het bealng van langlevenrisico geeft dit proefschrift verscheidene mogelijkheden om het langlevenrisico in hun portefeuille te kwantificeren en te reduceren. In de (nabije) toekomst, wanneer de babyboomers met pensioen gaan, waarbij pensioenfondsen vergrijzen en er daardoor minder mogelijkheden zijn om schokken in de dekkingsgraad op te vangen, zal de onzekerheid in de ontwikkeling van de overlevingskansen een belangrijke risicofactor zijn voor pensioenfondsen. Wellicht zullen er in de startende markt van langlevenrisico mogelijkheden zijn om het risico door te verkopen.

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